Numerical Analysis Qualifying Examination

June 2, 2006

NAME ____________________

SIGNATURE ____________________

#1	20	
#2	20	
#3	20	
Total	60	

Question 1. In this problem we will analyze the case of continuous piecewise *quadratic* interpolation on a mesh of $n+1$ knots $x_0 < x_1 < \ldots < x_n$. We will also need the interval midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$. The dimension of the space is $N = 2n + 1$.

a. Define the *nodal* basis functions. Note there are two types: *hat functions* and *bump functions*.

b. Let f^* be the continuous piecewise quadratic interpolant for f. Using the Peano Kernel Theorem. prove

$$\| f - f^* \|_\infty \leq Ch^3 \| f''' \|_\infty$$

Question 2. Let Let $y' = f(y)$, $y(0) = y_0$. Euler's method for solving this ordinary differential equation is given by

$$y_{k+1} = y_k + h\, f(y_k)$$

for $k = 0, 1, \ldots$ and $t_k = kh$. Let $T_f = nh$ denote the final time.

a. Compute the *local truncation error* for Euler's method.

b. Compute the *region of absolute stability* for Euler's method.

c. Using (discrete) Gronwall's Lemma, prove

$$\max_{0 \leq k \leq n} |y(t_k) - y_k| \leq Ch \max_{0 \leq t \leq T_f} |y''|$$

Question 3. The Euler-Maclaurin summation formula is

$$\int_a^b f(x)\, dx = T(h) + \sum_{k=1}^{r} C_k h^{2k} \{ f^{(2k-1)}(b) - f^{(2k-1)}(a) \} + O(h^{2r+2})$$

where $h = (b-a)/n$, $x_k = a + kh$, C_k is a constant independent of f and h, $f \in C^{2r-1}[a, b]$, and

$$T(h) = \frac{h}{2} \sum_{k=1}^{n} f(x_{k-1}) + f(x_k)$$

is the composite trapezoid rule. Using this information derive a Richardson Extrapolation scheme for computing a high order approximation of $\int_a^b f(x)\, dx$. Be sure to define all terms carefully and explicitly state the order of each intermediate approximation.

Numerical Analysis Qualifying Exam

Parts B and C

June 2, 2006

Name ______________________________

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
B-C	100	
A	60	
Total	160	

(20) 1. State and prove the *SVD* Existence Theorem (for real $m \times n$ matrices).

(20) 2. Let the *computed* L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E : $|E_{ij}| \leq (3+u)u\max(i-1, j)g$, $g = \max_k \max_{i,j} |a_{ij}^{(k)}|$.

(20) 3. Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$, where A is $m \times n$ and $m \geq n$, iff $\hat{x}$ satisfies the normal equations.

(20) 4. (a) Prove that if A is positive definite then its eigenvalues are positive.

(b) Prove that if A is normal and its eigenvalues are positive then A is positive definite.

(c) Prove that A is similar to a diagonal matrix iff A has n linearly independent eigenvectors, where A is $n \times n$.

(d) Prove that if A is real, then λ is a real eigenvalue of A iff it has a real corresponding eigenvector.

(20) 5. (a) State the Schur Decomposition Theorem.

(b) Use it to prove: if A is $n \times n$ then A has n orthonormal eigenvectors iff $A^H A = AA^H$.

(c) Show that if the single shift QR method converges, then the convergence is quadratic for general matrices.