Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Michael Holst

9:00am–12:00pm
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5829 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1.

(a) Let $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Prove that for all $1 \leq p \leq \infty$,

$$\|\Delta\|_p = \max_{1 \leq i \leq n} |\delta_i|.$$

(b) Let A and B be any pair of matrices such that the product AB is defined. Prove that $\|AB\|_F \leq \|A\|_2 \|B\|_F$.

(c) Let $\|\cdot\|$ and $\|\cdot\|_D$ denote any vector norm and its corresponding dual norm. If $A \in \mathbb{C}^{n \times n}$, let $\|A\|_D$ denote the matrix norm subordinate to $\|\cdot\|_D$. Prove that if x, $y \in \mathbb{C}^n$ then

$$\|xy^H\| = \|x\| \, \|y\|_D.$$

Question 1.2. Assume $A \in \mathbb{R}^{n \times n}$ is nonsingular. Find a solution E^* of the problem

$$\min_{E \in \mathbb{R}^{n \times n}} \left\{ \frac{\|E\|_2}{\|A\|_2} \mid A + E \ \text{ singular} \right\}.$$

Give the optimal value of $\|E\|_2 / \|A\|_2$. Comment on the uniqueness of E.

Question 1.3. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite.

(a) Prove that if Gaussian elimination without interchanges is applied to A, then the remaining matrix is symmetric positive definite at every step. Hence prove that there exist L and U such that $A = LU$.

(b) If A is factorized using Gaussian elimination without interchanges, prove that the growth bound satisfies $\rho_n \leq 1$.

2. Least-Squares and Eigenvalues

Question 2.1. Assume that $A \in \mathbb{R}^{m\times n}$.

(a) Prove that every $y \in \mathbb{R}^m$ has a unique decomposition $y = y_R + y_N$, with $y_R \in \text{range}(A)$ and $y_N \in \text{null}(A^T)$.

(b) For any x, let r denote the residual vector $b - Ax$. Prove that x solves the least-squares problem $\min \|b - Ax\|_2$ if and only if $Ax = b_R$ and $r = b_N$.

(c) Assume that A has full column rank.

  (i) Prove that the least-squares solution is unique.

  (ii) Describe how you would compute the least-squares solution using the QR decomposition of A.

Question 2.2. Consider a non-defective matrix $A \in \mathbb{C}^{2\times 2}$ such that

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$

(a) Find the left and right eigenvectors of A.

(b) Find the condition number of each of the eigenvalues of A.

Question 2.3. Let $A \in \mathbb{C}^{m\times n}$. Given an approximate eigenpair (λ, u), describe how you would use one step of inverse iteration to find an improved eigenvector v of A. Hence show that (λ, v) is an exact eigenpair of $A + E$ where E may be chosen to satisfy

$$\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.$$

3. Interpolation, Approximation and ODEs

Question 3.1. Let $f(x) \in C^\infty([a,b])$, let $p_n(x)$ interpolate $f(x)$ at the $n+1$ points $a = x_0 < x_1 < \cdots < x_n = b$, and denote $\mathcal{I}(f) = \int_a^b f(x)dx$.

(a) Show that the error in the interpolating polynomial can be written as:

$$e_n(x) = f(x) - p_n(x) = f[x_0, x_1, \ldots, x_n, x]\psi_n(x),$$

where $\psi_n(x) = \prod_{i=0}^n (x - x_i)$, and where $f[x_0, x_1, \ldots, x_n, x]$ is the divided difference at the $n+2$ points $x_0, x_1, \ldots, x_n, x$.

(b) Prove that

$$f[x_0, x_1, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}, \quad \xi \in (a,b).$$

(c) If $\int_a^b \psi_n(x)dx = 0$, show that

$$\mathcal{I}(e_n) = \mathcal{I}(f) - \mathcal{I}(p_n) = \frac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b \psi_n(x)dx, \quad \eta \in (a,b).$$

Question 3.2. Consider the problem of best L^2-approximation of $u \in C^\infty([a,b])$ from a subspace V: Find $u^* \in V$ such that

$$\|u - u^*\|_{L^2([a,b])} = \inf_{v \in V} \|u - v\|_{L^2([a,b])}.$$

(a) Derive a bound on the error in the best L^2-approximation of the form:

$$\|u - u^*\|_{L^2([a,b])} \leq Ch^3 \|u^{(3)}\|_{L^2([a,b])}, \quad h = b - a,$$

where u^* is chosen from the subspace of quadratic functions V.

(b) Given now the particular function $u(x) = x^3$ over the particular interval $[a,b] = [0,1]$, determine the best L^2-approximation from the subspace of quadratic functions, and justify the technique you use.