Numerical Analysis Qualifying Exam
June 5, 2000
Print Name ____________________
Signature ____________________

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|---|---|---|
| # A1 | 20 | |
| # A2 | 30 | |
| # A3 | 20 | |
| # B1 | 20 | |
| # B2 | 20 | |
| # B3 | 20 | |
| # B4 | 20 | |
| Subtotal | 150 | |
| #C | 50 | |
| Total | 200 | |

A1. (a) Prove $\|x\|_\infty \le \|x\|_p \le n^{\frac{1}{p}}\|x\|_\infty$ for all $x \in \mathbb{R}^n, 1 \le p \le \infty$.

(b) Let $A \in \mathbb{R}^{m\times n}$. Prove:

$\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$,

$\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$,

$\|A\|_2 \le \sqrt{\|A\|_1\ \|A\|_\infty}$.

A2. Let the *computed* $L$ and $U$ satisfy $A+E = LU$, where $L$ is unit lower triangular and $U$ is upper triangular. Derive the bound on $E : |E_{ij}| \le (3+u)u\max(i-1,j)g$, where $u$ is unit roundoff and $g = \max_{i,j,k} |a_{ij}^{(k)}|$.

A3. Prove that if $A$ is symmetric positive definite, $\max_{i,j}|a_{ij}| = 1$, then $\max_{i,j,k}|a_{ij}^{(k)}| = 1$ under $LU$ (or $LDL^T$) decomposition.

B1. Let $A \in \mathbb{C}^{n\times n}$. Prove that $A$ has $n$ orthonormal eigenvectors iff $A^HA = AA^H$.

B2. Let $A \in \mathbb{R}^{m\times n}, m \ge n$. Derive the min 2-norm least squares solution to $r = Ax - b$ in terms of the SVD of $A$.

B3. Let $A \in \mathbb{R}^{m\times n}, \text{rank}(A) = n, A^TAx = A^Tb, (A^TA+F)y = A^Tb, \|F\|_2 \le \frac{1}{2}\sigma_n(A)^2$, $r = b - Ax, s = b - Ay$. Show $s - r = A(A^TA+F)^{-1}Fx$ and $\|s-r\|_2 \le 2\kappa_2(A)\frac{\|F\|_2}{\|A\|_2}\|x\|_2$.

B4. Let $\tilde{A}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda\widetilde{B}\begin{bmatrix} y \\ z \end{bmatrix}$, where $\tilde{A} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}, \widetilde{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, A \in \mathbb{R}^{m\times n}$, $B_1 \in \mathbb{R}^{m\times m}, B_2 \in \mathbb{R}^{n\times n}, y \in \mathbb{R}^m, z \in \mathbb{R}^n, m \ge n$. Let $B_1, B_2$ be symmetric positive definite with Cholesky factors $G_1, G_2$. Relate the generalized eigenvalues of $(\tilde{A}, \widetilde{B})$ to the singular values of $M = G_1^{-1}AG_2^{-T}$.

*Numerical Analysis Qualifying Examination* **Part C:**

June 5, 2000

NAME ____________________

SIGNATURE ____________________

| | | |
|---|---|---|
| #1 | 25 | |
| #2 | 25 | |
| Total | 50 | |

**Question 1.** Let $f \in C^4(I)$, $I = [a,b]$, and let $x_i = a + ih$, $0 \le i \le n$, $h = (b-a)/n$ be a uniform mesh on $I$. Let $\mathcal{S}$ be the space of $C^1$ piecewise cubic hermite polynomials with respect to this uniform mesh and let $\tilde{f}$ denote the interpolant of $f$.

**a.** Compute the dimension of $\mathcal{S}$ and define the standard *nodal basis* functions for $\mathcal{S}$.

**b.** Using the Peano Kernel Theorem, prove:

$$\| f - \tilde{f} \|_{\mathcal{L}^2(I)} \le Ch^4 \| f^{(iv)} \|_{\mathcal{L}^4(I)}$$

(You do NOT need to explicitly evaluate the constant $C$.)

**Question 2.** Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

and let

$$\mathcal{Q}(f) = \sum_{i=1}^{n} w_i f(x_i)$$

denote the $n$-point Gauss-Legendre quadrature formula (of order $2n$). As usual denote by $\phi_i(x)$ the Lagrange nodal basis functions satisfying

$$\phi_i(x_j) = \delta_{ij}$$

**a.** Prove $w_i = \mathcal{I}(\phi_i)$.

**b.** Prove that the nodes $\{x_i\}$ are the zeroes of the Legendre polynomial of degree $n$. Hint: let $P(x)$ be a polynomial of degree $2n-1$ and $\tilde{P}(x) = \sum P(x_i)\phi_i(x)$ its Lagrange interpolant of degree $n$. First prove that $\mathcal{Q}(P) = \mathcal{I}(\tilde{P})$, and then consider the implications of $\mathcal{I}(P) = \mathcal{I}(\tilde{P})$ for *all* polynomials of degree $2n-1$.