Ph.D./Masters Qualifying Examination in Numerical Analysis

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10am–1pm
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5402 AP&M

NAME ____________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
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#3.1	30	
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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers, Linear Equations and Linear Least-Squares

In Parts 1 and 2, $\|\cdot\|_p$ refers to the vector p-norm or its subordinate matrix norm.

Question 1.1.

(a) Given any $x \in \mathbb{C}^m$, find positive constants c_1 and c_2, independent of x such that

$$c_1\|x\|_2 \le \|x\|_\infty \le c_2\|x\|_2.$$

(b) If $A \in \mathbb{C}^{m\times n}$, prove that $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A.

(c) Assume that $A \in \mathbb{C}^{m\times n}$ has rank r. Find a scalar σ ($\sigma > 0$), independent of p, such that

$$\|Ap\|_\infty \ge \sigma\|p\|_2 \quad \text{for all } p \in \text{range}(A^T).$$

Question 1.2.

(a) State the *standard rounding-error model* for floating-point arithmetic.

(b) Let u denote the unit round-off. Let n be a positive integer such that $n\mathrm{u} < 1$. If $\{\delta_i\}$ is a set of n numbers such that $|\delta_i| \le \mathrm{u}$, and $\{s_i\}$ are integers such that $s_i = \pm 1$, prove that

$$\prod_{i=1}^{n}(1+\delta_i)^{s_i} = 1 + \theta_n,$$

where $|\theta_n| \le \gamma_n$, with $\gamma_n = n\mathrm{u}/(1 - n\mathrm{u})$.

(c) Given two n-vectors x and y, let $\widehat{Z}$ denote the *computed* version of the rank-one matrix $Z = xy^T$. Apply the standard rounding error model to derive a bound for the component-wise forward error in $\widehat{Z}$ as an approximation to Z. Is the calculation of $\widehat{Z}$ backward stable? Justify your answer.

Question 1.3. Assume that $A \in \mathbb{R}^{n\times n}$.

(a) Suppose that r ($r < n$) steps of Householder reduction with column interchanges gives the decomposition

$$AP = Q\begin{pmatrix} R_{11} & R_{12} \\ & 0 \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r \times r$ nonsingular upper triangle. Define bases for null(A) and range(A^T) in terms of the QR factors above. Verify that the proposed bases satisfy the properties of a basis.

(b) Now assume that r steps of Householder reduction give:

$$AP = Q\begin{pmatrix} R_{11} & R_{12} \\ & E \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r\times r$ nonsingular upper triangle. Show that σ_n, the smallest singular value of A, satisfies $\sigma_n \le \|E\|_2$. Give a *brief* discussion of the implication of this result.

2. Nonlinear Equations, Nonlinear Least-Squares and Optimization

Question 2.1.

(a) Let $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be continuously differentiable on the open convex set $\mathcal{D}$. Compute the Fréchet derivative for the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $f(x) = \|x\|_2$.

(b) Given a real $n \times n$ symmetric matrix A, find the Fréchet derivative of the function $G : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$ such that

$$G(x, \lambda) = \begin{pmatrix} Ax - \lambda x \\ \|x\|_2 - 1 \end{pmatrix}.$$

Hence define an iteration of Newton's method for finding an eigenvalue of A and its associated eigenvector.

(c) An eigenvalue of a matrix is *simple* if it has algebraic multiplicity 1. If λ^* is a simple eigenvalue of A and x^* is its corresponding normalized eigenvector, prove that $G'(x^*, \lambda^*)$ is nonsingular. Give a *brief* discussion of the implication of this result when finding x^* and λ^* using Newton's method.

Question 2.2. Consider the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

(a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0,\ 1,\ 0)^T$.

(b) Compute the Newton direction p^N and modified Newton direction p^M at $\bar{x}$. Determine if p^N and p^M are descent directions.

(c) Find a direction of negative curvature that is a direction of decrease for f at $\bar{x}$.

Question 2.3.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where γ is a scalar and u and v are n vectors.

(b) Given an $n \times n$ symmetric positive-definite matrix B, and n-vectors y and s, consider the symmetric rank-one quasi-Newton update

$$B_+ = B + \frac{1}{(y - Bs)^T s}(y - Bs)(y - Bs)^T. \tag{2.1}$$

(i) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a quadratic function with a symmetric positive-definite Hessian matrix. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$, where $\nabla f(x)$ is the gradient of f evaluated at x. If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, prove that $B_+\bar{s} = \bar{y}$.

(ii) Find a condition on the vectors y and s that will guarantee the positive definiteness of B_+.

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3. Approximation and Numerical ODEs

In this part, we assume that a, $b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Prove for any $f \in C[a,b]$ that

$$\lim_{n\to\infty} \inf_{q_n\in\mathcal{P}_n} \max_{a\leq x\leq b} |f(x) - q_n(x)| = 0,$$

$$\lim_{n\to\infty} \inf_{q_n\in\mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2]\, dx = 0.$$

(b) Let $p_2 \in \mathcal{P}_2$ be the best uniform approximation in $\mathcal{P}_2$ of the function $g(x) = x^3 - 2x^2 + 1$ with respect to the $C[-1,1]$-norm. What is the value of $p_2(1)$? Justify your answer.

(c) Let $Q_0, \ldots, Q_n, \ldots$ be orthogonal polynomials in $L^2[a,b]$. Fix $n \geq 1$. Prove that Q_n has n simple roots in $[a,b]$.

Question 3.2.

(a) Find the degree of precision of the numerical quadrature

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)] \qquad \forall f \in C^1[a,b].$$

(b) Consider a sequence of interpolatory numerical integration formulas

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}), \qquad n = 1, \ldots.$$

Suppose all the coefficients $A_k^{(n)}$ $(k = 1, \cdots, n;\ n = 1, \cdots)$ are positive. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) = \int_a^b f(x)\, dx \qquad \forall f \in C[a,b].$$