Math 270abc: Numerical Analysis

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Qualifying Examination
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Question 1. Let

$$A = \frac{1}{10}\begin{bmatrix} -30 & -10 & -20 \\ -60 & -40 & -50 \\ 91 & 70 & 80 \end{bmatrix}, \qquad A^{-1} = \frac{1}{3}\begin{bmatrix} -30 & 60 & 30 \\ -25 & 58 & 30 \\ 56 & -119 & -60 \end{bmatrix},$$

and given $\vec{b}, \delta\vec{b} \in \mathbb{R}^3$, let $\vec{x} \in \mathbb{R}^3$ satisfy the linear system $A\vec{x} = \vec{b}$, and $\vec{x} + \delta\vec{x} \in \mathbb{R}^3$ the perturbed linear system $A(\vec{x} + \delta\vec{x}) = \vec{b} + \delta\vec{b}$.

Find one example of $\vec{b} \neq 0$ and $\delta\vec{b}$ where the relative perturbation satisfies $\frac{||\delta\vec{b}||_1}{||\vec{b}||_1} = \frac{1}{1000}$ and the relative error, $\frac{||\delta\vec{x}||_1}{||\vec{x}||_1}$, is maximized.

Recall: $||x||_1 = \sum_{i=1}^{n} |x_i|$ and $||A||_1 = \max_{\vec{x}\neq 0} \frac{||A\vec{x}||_1}{||\vec{x}||_1} = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$.

Question 2. Consider the linear system $A\vec{x} = \vec{b}$, with $\vec{x}, \vec{b} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ satisfying $a_{ii} \neq 0$, for any $i = 1, \ldots, n$. Furthermore, suppose the Jacobi iterative method on $A\vec{x} = \vec{b}$ converges for all initial vectors, and its iteration matrix, B_J, has all real eigenvalues.

Now consider the following iterative method, with parameter $\alpha \in \mathbb{R}, \alpha \neq 0$, defined by the iteration formula

$$\vec{x}^{(k+1)} = M_\alpha^{-1} N_\alpha \vec{x}^{(k)} + M_\alpha^{-1}\vec{b},$$

where $A = M_\alpha - N_\alpha$ and $M_\alpha = \alpha D \in \mathbb{R}^{n\times n}, N_\alpha \in \mathbb{R}^{n\times n}$, with $D \in \mathbb{R}^{n\times n}$ the diagonal matrix satisfying $d_{ii} = a_{ii}$ for $i = 1, \ldots, n$.

Derive an expression, in terms of eigenvalues of B_J, for the parameter α that maximizes the (asymptotic) rate of convergence of this method.

Question 3. Let an *RQ* factorization of $A \in \mathbb{R}^{m\times n}$, if it exists, refer to the matrices $R \in \mathbb{R}^{m\times m}$, upper triangular, and $Q \in \mathbb{R}^{m\times n}$, with orthonormal rows, satisfying $A = RQ$.

- **a.** Given $A \in \mathbb{R}^{3\times 5}$, with linearly independent rows, write down, step-by-step, a method for calculating the RQ factorization of A, and explain the purpose of each step.
- **b.** Suppose $A \in \mathbb{R}^{m\times n}$ has linearly independent rows. Given $\vec{b} \in \mathbb{R}^m$ and an RQ factorization of A, derive an expression, in terms of R, Q, and $\vec{b}$, for the (least squares solution) $\vec{x} \in \mathbb{R}^n$ with the smallest 2-norm satisfying $A\vec{x} = \vec{b}$.

Question 4. Let $f(x) \in \mathcal{C}^2$ be a scalar function of a scalar variable x.

- **a.** Define Newton's method and the Secant method for solving $f(x^*) = 0$.
- **b.** Prove Newton's methods is locally quadratically convergent. Define "locally" in this context, and be sure to carefully list your assumptions.
- **c.** Prove the Secant method is locally superlinearly convergent with rate $\rho = (1 + \sqrt{5})/2$. Be sure to carefully list your assumptions.

Question 5. In this problem we will analyze the case of continuous piecewise linear interpolation of $f \in C^2[a, b]$ on a mesh of $n + 1$ knots $a = x_0 < x_1 < \cdots < x_n = b$, with $h_i = x_i - x_{i-1}$, $h = max_i h_i$.

- **a.** Define the interpolant f^* of f on the interval $[x_{i-1}, x_i]$.
- **b.** Derive the Peano Kernel $K(x, t)$ for this approximation on $[x_{i-1}, x_i]$ given by

$$K(x,t) = \frac{-1}{h_i}\begin{cases} (x - x_i)(t - x_{i-1}) & x_{i-1} \leq x \leq t \\ (t - x_i)(x - x_{i-1}) & t \leq x \leq x_i \end{cases}$$

c. Prove, using the Peano Kernel Theorem,

$$\|f - f^*\|_{L_\infty[a,b]} \leq \frac{h^2}{8}\|f''\|_{L_\infty[a,b]}$$

Question 6. Consider the inner product with weight function $w = 1 - x^2$

$$(f, g) = \int_{-1}^{1} fgw\,dx.$$

a. Compute the the 2-point Gaussian Quadrature formula $Q(f)$ to approximate the integral

$$I(f) = \int_{-1}^{1} fw\,dx.$$

b. Prove an error bound for $|I(f) - Q(f)|$. Be sure to explicitly evaluate the constant, and state all of your assumptions.

Question 7. Consider the following one-parameter family of explicit, 2-stage Runge–Kutta methods, where $\alpha \neq 0$,

$$\begin{aligned} Y^1 &= y_n, \\ Y^2 &= y_n + \alpha h f(t_n, Y^1), \\ y_{n+1} &= y_n + \left(1 - \frac{1}{2\alpha}\right) h f(t_n, Y^1) + \frac{1}{2\alpha} h f(t_n + \alpha h, Y^2). \end{aligned}$$

a. Determine if this Runge–Kutta method is a collocation Runge–Kutta method, for all values of α. Explain how you arrived at your conclusion.

b. Determine the order of accuracy of this method, for all values of α.

c. Prove that the stability region of this method is independent of α.

Question 8. Consider a linear multistep method of the form,

$$y_{n+4} = y_{n+3} + h\left[b_1 f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n)\right].$$

a. The Adams methods are based on interpolation and the fundamental theorem of calculus. By following that approach, derive the values of b_0 and b_1 in the formula above. Does the method you derived have the highest order that can be achieved for a method of that general form? Explain your reasoning.

b. Explain how the root condition for the stability of linear multistep methods is related to the solution theory for linear constant coefficient homogeneous difference equations.

c. Determine if the linear multistep method you derived is convergent, and explain how you arrived at your conclusion.