Ph.D./Masters Qualifying Examination in Numerical Analysis

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9am–12pm
Wednesday September 9, 2015

Name ______________________

#1.1	25	
#1.2	25	
#1.3	25	
#2.1	25	
#2.2	25	
#2.3	25	
#3.1	25	
#3.2	25	
Total	200	

- Put your name in box provided and staple page to your solutions.
- Write your name clearly on every sheet submitted.

1 Numerical Linear Algebra (270A)

Question 1.1. Recall two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ on V on a normed space V are *equivalent* if:

$$C_1\|u\|_X \leq \|u\|_Y \leq C_2\|u\|_X, \quad \forall u \in V.$$

(a) Consider now $V = \mathbb{R}^n$, and show the following norm equivalence bounds:

$$\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1 \leq \sqrt{n}\|u\|_2 \leq n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n,$$

and then show the following matrix norm equivalence relationships:

$$\|A\|_1 \leq \sqrt{n}\|A\|_2 \leq n\|A\|_1, \qquad \|A\|_\infty \leq \sqrt{n}\|A\|_2 \leq n\|A\|_\infty.$$

(b) Assume A is invertible, and derive analogous equivalence relationships for condition numbers.

(c) Show that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, if $\rho(A)$ is the spectral radius of $A \in \mathbb{R}^{n\times n}$, then

$$\rho(A) \leq \|A\|.$$

Question 1.2. Let $P \in \mathbb{R}^{n\times n}$ be a projector. Recall that this means P is idempotent ($P^2 = P$). Recall also that to be an *orthogonal projector*, P would also have to be self-adjoint ($P = P^*$), and it would not mean that P is an orthogonal matrix.

(a) Show that if P is a non-zero projector, then $\|P\|_2 \geq 1$, with equality iff P is self-adjoint.

Consider the overdetermined system: $Ax = b, \quad A \in \mathbb{R}^{m\times n}, \quad m \geq n.$

(b) Show that A^TA is nonsingular if and only if A has full rank.

Consider now the particular instance of this overdetermined system:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}. \tag{1.1}$$

(c) Before calculating, look at equations and predict what least-squares solution will be.

(d) Calculate a QR decomposition, where Q is a 2×2 rotator (or reflector), and R is 2×1.

(e) Use the QR decomposition to calculate the least squares solution to (1.1).

(f) Instead of QR, form and solve normal equations to produce the least squares solution to (1.1).

(g) Give expression for pseudo-inverse A^+, then give instance you formed solving problem above.

Question 1.3. For this problem, assume $A \in \mathbb{R}^{n\times n}$ is given and that A is invertible.

(a) Give precise definition of *condition* of a general mathematical problem, and apply definition to specific problem $Ax = b$. First assume b is given and you must find x, and then assume x is given and you must find b. Show how the condition number arises naturally in both cases.

(b) Give precise definitions of the *stability* and *backward stability* of an algorithm, and explain the distinction between these two properties of an algorithm and an algorithm being *unstable*.

(c) Given A and b, with error δb in b, show that the error in solving $Ax = b$ obeys the bound:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|},$$

where $Ax = b$, and where $A(x + \delta x) = (b + \delta b)$.

2 Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Given V a Banach space and $T : K \to K$ a contractive mapping satisfying $T(0) = 0$, for $K = \{v \in V \mid ||v|| \leq 1\}$, show there exists $r > 0$ such that $u = T(u) + y$ has a unique solution in K for all $y \in \{v \in V \mid ||v|| \leq r\}$.

Question 2.2. Given distinct nodes $x_i \in \mathbf{R}, i = 0, \dots, n$, and given values $y_i, z_i \in \mathbf{R}, i = 0, \dots, n$, prove the uniqueness of the Hermite interpolating polynomial: the polynomial p of degree $\leq 2n+1$ satisfying $p(x_i) = y_i, p'(x_i) = z_i, i = 0, \dots, n$.

Question 2.3. Given $f \in C^2(\mathbf{R})$, show the midpoint rule satisfies

$$\int_0^{2h} f(x)\ dx - 2hf(h) = \int_0^{2h} K(x) f''(x)\ dx,$$

where

$$K(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x < h \\ \frac{(x-2h)^2}{2}, & h \leq x \leq 2h. \end{cases}$$

3 Numerical Ordinary Differential Equations (270C)

Question 3.1. Consider the initial value problem,

$$y' = f(t, y), \qquad y(0) = y_0.$$

1. Derive a one-parameter family of explicit, two-stage, second-order Runge–Kutta method of the approximate solution of this problem of the form,

   $$y_{n+1} = y_n + h[\alpha f(t_n, y_n) + \beta f(t_n + \theta h, y_n + k_n)],$$

   giving expressions for α, β, θ, and k_n in terms of a parameter λ. Justify your answer.
2. Write down the Butcher tableau for your method.

Question 3.2. Consider the initial value problem,

$$y' = f(t, y), \qquad y(0) = y_0,$$

and a linear multistep method of the following form,

$$y_{n+1} = \alpha y_n + \beta y_{n-1} + h\gamma f(t_{n-1}, y_{n-1}).$$

1. Choose the constants α, β, and γ, so that the order of the method is as high as possible.
2. Determine whether the resulting method is convergent, and justify your answer.