NA Qual: Parts A and B

Sept. 9. 2008

Name ______________________

#1	30	
#2	35	
#3	50	
#4	35	
A-B	160	
Part C	40	
Total	200	

1. (a) (10) Prove that if $T \in \mathbb{C}^{n\times n}$,

$$T = \begin{array}{cc} \left[\begin{array}{cc} T_{11} & T_{12} \\ 0 & T_{22} \end{array}\right] & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & \quad q \end{array} & \end{array},$$

then $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$.

   (b) (20) Prove that if $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{p\times p}$, and $X \in \mathbb{C}^{n\times p}$ satisfy $AX = XB$, rank$(X) = p$, then there exists a unitary $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^H AQ = T = \begin{array}{cc} \left[\begin{array}{cc} T_{11} & T_{12} \\ 0 & T_{22} \end{array}\right] & \begin{array}{c} p \\ n-p \end{array} \\ \begin{array}{cc} p & \quad n-p \end{array} & \end{array},$$

where $\lambda(T_{11}) = \lambda(A) \cup \lambda(B)$. (Hint: consider QR decomposition of X, and use (a).)

2. (a) (25) State and prove the Schur Decomposition Theorem for $A \in \mathbb{C}^{n\times n}$. (Hint: use induction and 1(b).)

   (b) (10) Use 2(a) to prove that $A \in \mathbb{C}^{n\times n}$ has n orthonormal eigenvectors iff $A^H A = AA^H$.

3. (a) (25) State and prove the SVD Existence Theorem for $A \in \mathbb{R}^{m\times n}$.

   (b) (15) Let $A \in \mathbb{R}^{m\times n}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{11} is $k \times k$. Use $S = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$, $m \times n$, to show that $\sigma_{k+1}(A) \leq \|A_{22}\|_2$.

   (c) (10) Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$. Show that there exist an orthogonal Q and a symmetric positive semi-definite P such that $A = QP$.

4. (a) (20) Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$ iff $\hat{x}$ satisfies the normal equations, where A is $m \times n$, $m \geq n$.

   (b) (15) Let A by $n \times n$, symmetric positive definite. Let $u_1, \ldots, u_n$ be an orthonormal basis of eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$. Let $w = \sum_{j=1}^{n} \alpha_j u_j$, $S(\mu) \equiv (A + \mu I)^{-1} w$, $\mu > 0$. Show that

$$\frac{d}{d\mu}\|S(\mu)\|_2 = -\frac{S(\mu)^T (A + \mu I)^{-1} S(\mu)}{\|S(\mu)\|_2}.$$

Numerical Analysis Qualifying Examination
9:00–12:00, Tuesday, September 9, 2008, AP&M 5829

Part C: Approximation, Interpolation, and Numerical Quadrature. We assume that $a, b \in \mathbb{R}$ with $a < b$. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ and by $\overline{\mathcal{P}}_n$ the set of all polynomials in $\mathcal{P}_n$ with leading coefficient 1.

Question 3.1 [20 points]

Let $n \geq 1$ be an integer. Let $Q_k \in \overline{\mathcal{P}}_k$ $(k = 0, \ldots, n)$ be such that

$$\int_a^b Q_j(x) Q_k(x)\, dx = 0$$

for any indices j and k with $0 \leq j < k \leq n$. Let $P_n \in \overline{\mathcal{P}}_n$. Prove the following:

(a) The identity

$$P_n(x) = c_0 Q_0(x) + \cdots + c_{n-1} Q_{n-1}(x) + Q_n(x)$$

holds true for a unique set of real numbers $c_0, c_1, \ldots, c_{n-1}$. Moreover,

$$\int_a^b |P_n(x)|^2 dx = c_0^2 \int_a^b |Q_0(x)|^2 dx + \cdots + c_{n-1}^2 \int_a^b |Q_{n-1}(x)|^2 dx + \int_a^b |Q_n(x)|^2 dx;$$

(b) The inequality

$$\int_a^b |Q_n(x)|^2 dx \leq \int_a^b |P_n(x)|^2 dx$$

holds true. Moreover, this inequality becomes equality if and only if $P_n = Q_n$.

Question 3.2 [20 points]

(1) Calculate the Lagrange interpolation polynomial that interpolates the function $f(x) = x^{10}$ at points $x = 0, 1, \ldots, 20$. Justify your answer.

(2) Let $N \geq 1$ be an integer, $h = (b - a)/N$, and $x_j = a + jh$ $(j = 0, \ldots, N)$. The composite mid-point quadrature is given by

$$\int_a^b f(x)\, dx \approx h \sum_{j=1}^{N} f\left(\frac{x_{j-1} + x_j}{2}\right).$$

Suppose $f \in C^2[a, b]$. Prove that there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, dx - h \sum_{j=1}^{N} f\left(\frac{x_{j-1} + x_j}{2}\right) = \frac{1}{24}(b - a) h^2 f''(\xi).$$