Numerical Analysis Qualifying Examination

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NAME ____________________

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Total	150	

Question 1. Let A be a $n \times n$ nonsingular matrix. Prove by induction that $PA = LU$, where P is a permutation matrix, L is unit lower triangular, and U is upper triangular.

Question 2. Let A and B be symmetric, positive definite $N \times N$ matrices. Assume there exist positive constants α and β such that

$$\alpha \leq \frac{x^t A x}{x^t B x} \leq \beta$$

for all $x \neq 0$. Consider the solution of $Ax = b$ by the iterative method:

$$B(x_{k+1} - x_k) = \omega(b - Ax_k)$$

where x_0 is given and $\omega = 2/(\alpha + \beta)$.

a. Derive the error propagator G for this iteration.

b. Prove:

$$\|e_k\|_A \leq \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^k \|e_0\|_A$$

where $e_k = x - x_k$.

Question 3. Let $\phi(\vec{x})$ be a scalar function of the vector variable $\vec{x}$. Suppose $\phi(\vec{x})$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite.

1. Formally define Newton's method with line search for solving the optimization problem $\min_{\vec{x}} \phi(\vec{x})$.
2. Let $\vec{p}_k$ be the Newton search direction. Show that $\partial\phi(\vec{x}_k + \alpha\vec{p}_k)/\partial\alpha < 0$ at $\alpha = 0$. Why is this fact significant for the line search?

Question 4. Consider the fundamental quadrature formula (Simpson's Rule)

$$\mathcal{I}(f) = \int_0^1 f(x)\,dx, \qquad \mathcal{Q}(f) = \frac{f(0) + 4f(1/2) + f(1)}{6}.$$

a. Using the Peano Kernel Theorem, prove

$$|\mathcal{E}(f)| = |\mathcal{I}(f) - \mathcal{Q}(f)| \leq \frac{\|f^{iv}\|_{\infty[0,1]}}{2880}.$$

You may prove this result for generic C rather than $C = 1/2880$.

b. Derive the *Composite* Simpson's Rule $\mathcal{Q}_c(f)$ for approximating

$$\int_a^b f(x)\,dx$$

with a uniform mesh of size h.

c. Using part **a**, prove

$$\left| \int_a^b f(x)\,dx - \mathcal{Q}_c(f) \right| \leq \frac{h^4\,|b-a|\,\|f^{iv}\|_{\infty[a,b]}}{2880}.$$

Question 5. Consider the initial value problem:

$$
\begin{aligned}
y' &= f(y) \\
y(x_0) &= y_0
\end{aligned}
$$

and the consider the second backward difference formula:

$$y_{k+1} = \alpha_1 y_k + \alpha_2 y_{k-1} + h\beta_0 f(y_{k+1})$$

a. Find α_1, α_2 and β_0 to maximize the order.

b. Find the local truncation error.

c. Find the region of absolute stability for the method. Is the method A-stable? L-stable?

Question 6. Consider the 2-point boundary value problem $-u'' + u = f$, $u(0) = u(1) = 0$, and its Ritz formulation: Find $u \in H_0^1$ such that

$$a(u,u) - 2\,(f,u) = \min_{v \in H_0^1} a(v,v) - 2\,(f,v)$$

$$a(u,v) = \int_0^1 u'v' + uv\,dx \qquad\qquad (f,v) = \int_0^1 fv\,dx$$

$$\|u\|^2 = a(u,u) \qquad\qquad |u|^2 = (u,u)$$

Let $S_0 \subset H_0^1$ be the space of continuous piecewise linear polynomials and let $u_h \in S_0$ be the finite element approximation satisfying

$$a(u_h,u_h) - 2\,(f,u_h) = \min_{v \in S_0} a(v,v) - 2\,(f,v).$$

a. Prove the Ritz formulation is equivalent to the Galerkin formulation: Find $u \in H_0^1$ such that

$$a(u,v) = (f,v)$$

for all $v \in H_0^1$. A similar result holds for u_h; you may assume that result and need not prove it.

b. Using part **a**, prove the best approximation result

$$\|u - u_h\| = \min_{v \in S_0} \|u - v\|.$$

Hint: the important step is to show u_h is an orthogonal projection, $a(u - u_h, v) = 0$ for all $v \in S_0$.