Ph.D./Masters Qualifying Examination in Numerical Analysis

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10:00am–1:00pm
Monday September 10, 2001
6438 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1. Assume that $A \in C^{m\times n}$.

(a) Define the one-norm $\|A\|_1$, two-norm $\|A\|_2$, and infinity norm $\|A\|_\infty$ of A. Show that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|.$$

(b) Establish the following identities between the one-norm, two-norm and infinity-norm:

(i) $\|A\|_1 \le n\,\|A\|_\infty$.

(ii) $\dfrac{1}{m}\,\|A\|_\infty \le \|A\|_1$.

(iii) $\|A\|_2^2 \le \|A\|_1\,\|A\|_\infty$.

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \ne b$. Suppose that $\tilde{a}$ and $\tilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error in $\tilde{x} = \tilde{a} - \tilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the calculation $s = ab+c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3. Let $A \in I\!R^{n\times n}$ be symmetric positive definite.

(a) Prove that

$$\begin{aligned} a_{ii} &> 0 \quad \text{for all } i, \\ |a_{ij}| &\le \sqrt{a_{ii}a_{jj}} \quad \text{for all } i \text{ and } j, \\ \max_{i,j} |a_{ij}| &= \max_{i} a_{ii} \end{aligned}$$

(b) Prove that every symmetric positive-definite matrix A can be factorized as $A = LDL^T$, where L is unit lower-triangular and D is a diagonal matrix with positive diagonals.

2. Least-Squares and Eigenvalues

Question 2.1. Assume that $A \in \mathbb{R}^{m\times n}$ and $b \in \text{range}(A)$.

(a) State and prove the necessary and sufficient conditions for a vector x_+ to be a solution of minimum two-norm for the compatible system $Ax = b$.

(b) Show that the least-length solution x_+ of $Ax = b$ is unique.

(c) Assume that A has rank m. Given an QR factorization of A^T, describe how to compute each of the following quantities.

- **(i)** A basis for null(A).
- **(ii)** The general solution of $Ax = b$.
- **(iii)** The minimum-length solution of $Ax = b$.
- **(iv)** The pseudoinverse of A.

In cases (i) and (iii), prove that the computed quantity has the required properties.

Question 2.2. Assume $A \in C^{n\times n}$ is nondefective, with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Consider the perturbation $\widetilde{A} = A + \epsilon F$, with $|F_{ij}| \leq 1$. Show that the eigenvalues of $\widetilde{A}$ lie in circular discs with centers $\lambda_i + \epsilon\phi_{ii}/s_i$ and radii $\epsilon\sum_{j\neq i}|\phi_{ij}/s_i|$, where $s_i = y^H{}_i x_i$ and $\phi_{ij} = y^H{}_i F x_j$. Hence show that the ith disc must have radius less than $n(n-1)\epsilon/|s_i|$.

Question 2.3.

(a) Briefly describe the *explicitly* shifted QR method for finding the Schur decomposition of a matrix A.

(b) Let Q_k and R_k denote the matrices associated with the kth step of the explicitly shifted QR method. If $\widehat{Q}_k = Q_0Q_1\cdots Q_{k-1}Q_k$ and $\widehat{R}_k = R_kR_{k-1}\cdots R_1R_0$, establish the identities:

- **(i)** $H_{k+1} = Q_k^H H_k Q_k$.
- **(ii)** $H_{k+1} = \widehat{Q}_k^H H \widehat{Q}_k$.
- **(iii)** $\widehat{Q}_k\widehat{R}_k = \Pi_{j=0}^{k}(H - \mu_j I)$.

3. Interpolation, Approximation and ODEs

Question 3.1. Let $f(x) \in C^{\infty}([a,b])$ and let $\{x_k\}_{k=0}^{n}$ be $n+1$ distinct points in (a,b) such that $a = x_0 < x_1 < \cdots < x_n = b$.

(a) Give a constructive proof that there exists a unique polynomial $p_n(x)$ which interpolates $f(x)$ at the $n+1$ points $\{x_k\}_{k=0}^{n}$.

(b) Prove that the error in the interpolating polynomial can be written as:

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\psi_n(x),$$

for some $\xi \in (a,b)$, where $\psi_n(x) = \prod_{i=0}^{n}(x - x_i)$.

(c) Prove that the k-th divided difference $p[x_0, \ldots, x_k]$ of a polynomial $p(x)$ of degree $\leq k$ is independent of the interpolation points $x_0, x_1, \ldots, x_k$.

Question 3.2.

(a) Let $P_n(x)$ be an orthogonal family of polynomials, and let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of P_{k+1}, the $(k+1)$-st polynomial in the family. Prove that the Lagrange polynomials built from these points:

$$L_{k,i}(x) = \prod_{j=0, j\neq i}^{k} \frac{(x - x_j)}{(x_i - x_j)}, \quad i = 0, \ldots, k$$

are mutually orthogonal. Use this result to show that the Gauss weights

$$A_i = \mathcal{I}[L_{k,i}(x)w(x)],$$

based on this family are always positive, where $w(x)$ is a positive weight function on $[a,b]$.

(b) Use a two-point Gaussian quadrature rule, together with a coordinate transformation if necessary, to approximate the following integral:

$$\int_0^{10} e^{-x^2}\, dx.$$

(c) Give a (reasonably tight) upper-bound on the number of integration intervals n required to guarantee that the composite Simpson rule will approximate the integral in Part (c) to six digits of accuracy.