Numerical Analysis Qualifying Exam
September 12, 2000

Name ______________________

# A1	25	
# A2	10	
# A3	25	
# A4	25	
# B1	15	
# B2	30	
# B3	20	
Subtotal	150	
#C	50	
Total	200	

(25) A1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(10) A2. **Def:** $|||\ \ |||$ is a **subordinate** matrix norm $\equiv$ there exists a vector norm $\|\ \ \|$ such that $|||A||| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ for all A.

(a) Prove that if $|||\ \ |||$ is a subordinate matrix norm then $|||I||| = 1$.

(b) Show that the Frobenius norm is not subordinate to any vector norm.

(25) A3. (a) Let A be $m \times n, m > n, B = [A|z]$. Show that $\sigma_1(B) \geq \sigma_1(A)$ and $\sigma_{n+1}(B) \leq \sigma_n(A)$.

(b) Let A be $m \times n, m \geq n, C = \begin{bmatrix} A \\ v^T \end{bmatrix}$. Show that $\sigma_n(C) \geq \sigma_n(A)$ and $\sigma_1(A) \leq \sigma_1(C) \leq \sqrt{\sigma_1(A)^2 + v^T v}$.

(25) A4. Let the *computed* L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on $E : |E_{ij}| \leq (3 + u)u \max(i - 1, j)g$, where u is unit roundoff and $g = \max_{i,j,k} |a_{ij}^{(k)}|$.

(15) B1. Use Gershgorin's Theorem to prove that a real symmetric diagonally dominant matrix with positive diagonal elements is positive definite.

(30) B2. (a) Let $r = Ax - b, A$ is $m \times n, m \geq n, \text{rank}(A) = k < n$. Derive the min 2-norm least squares solution of $r = Ax - b$ in terms of the SVD of A.

(b) Let $Ax = b, A$ is $m \times n, m < n, \text{rank}(A, b) = \text{rank}(A) = k$. Derive the min 2-norm solution to $Ax = b$ in terms of the SVD of A.

(20) B3. Show that if the single shift QR method converges, then the convergence is:

(a) quadratic for general matrices,

(b) cubic for symmetric matrices.

Numerical Analysis Qualifying Examination

Part C

September 12, 2000

NAME ________________

SIGNATURE ________________

#1	25	
#2	25	
Total	50	

Question 1. Consider the initial value problem:

$$\begin{aligned} y' &= f(y) \\ y(x_0) &= y_0 \end{aligned}$$

and the following three methods for approximating its solution:

$$\begin{aligned} y_{n+1} &= y_n + hf(y_n) \\ y_{n+1} &= y_n + hf(y_{n+1}) \\ y_{n+1} &= y_{n-1} + 2hf(y_n) \end{aligned}$$

a. For each method, compute the local truncation error and the resulting order of approximation.

b. For each method, compute the region of absolute stability.

Question 2. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

Consider the two point Gauss-Legendre quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2) \tag{1}$$

a. Find the knots x_1 and x_2 and the weights w_1 and w_2 for the Gauss-Legendre formula (1).

b. Derive an error estimate for $\mathcal{E}(f) = |\mathcal{I}(f) - \mathcal{Q}(f)|$. Be sure to explicitly evaluate the constant.

c. Derive the composite formula for approximating

$$\int_a^b f(x)dx$$

on a uniform mesh of size h (note here the reference interval is $[-1, 1]$).

d. Write down an expression for the error in the composite formula. (No proof needed.)