Math 270abc: Numerical Analysis

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Qualifying Examination
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Question 1. Suppose $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $A \in \mathbb{R}^{n \times n}$, satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|,$$

and let $q_1 \in \mathbb{R}^n$, with $||q_1||_2 = 1$, be an eigenvector corresponding to the eigenvalue λ_1. Suppose there exists orthogonal $Q \in \mathbb{R}^{n \times n}$, with q_1 as its first column, and an upper triangular $U = (u_{ij}) \in \mathbb{R}^{n \times n}$, with $u_{11} = \lambda_1$, such that $Q^T A Q = U$. Given $x \in \mathbb{R}^n$, prove that there exists $\beta \in \mathbb{R}$ such that

$$\lim_{k \to \infty} \frac{A^k x}{\lambda_1^k} = \beta q_1.$$

Note: $(I - C)(I + C + \cdots + C^{k-1}) = I - C^k$, for any square matrix C.

Question 2. For this problem, ignore overflow and underflow. Consider the linear system $Ax = b$, and suppose $b \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

is nonsingular (note the real-valued entries). Suppose this linear system is solved using forward substitution in a machine with unit roundoff error $u < 1$, giving the solution $\hat{x}$ (with machine-valued entries). Show $\hat{x}$ satisfies $C\hat{x} = b$, for some lower triangular $C = (c_{ij}) \in \mathbb{R}^{3 \times 3}$ with $c_{31} = 0$ and

$$|c_{33} - a_{33}| \leq (4u + \mathcal{O}(u^2))|a_{33}|.$$

Question 3. Given $m \geq n$ and linearly independent $a_1, \dots, a_n \in \mathbb{R}^m$, let $q_1 = a_1$ and

$$q_k = \left(I - \frac{q_{k-1} q_{k-1}^T}{q_{k-1}^T q_{k-1}} \right) \cdot \ldots \cdot \left(I - \frac{q_1 q_1^T}{q_1^T q_1} \right) a_k,$$

for $2 \leq k \leq n$. Define $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ and $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \in \mathbb{R}^{m \times n}$.

(a) Prove $Q^T Q \in \mathbb{R}^{n \times n}$ is diagonal.

(b) Prove $A = QR$ for some unit upper triangular $R \in \mathbb{R}^{n \times n}$.

Question 4. Let $\phi(x)$ be a scalar function of the n-vector variable x. Suppose $\phi(x)$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite. Let A be an $m \times n$ matrix, $n > m$ of full rank. Consider the equality constrained optimization problem

$$\min_{Ax=b} \phi(x)$$

(a) Formally define the Lagrangian for this problem.

(b) State the necessary conditions for existence and uniqueness of a solution.

(c) Derive Newton's Method (KKT system) for solving this problem.

Question 5. Suppose we wish to approximate the function $f(x) = e^x$ on the interval $0 \leq x \leq 10$ using continuous piecewise linear interpolation on a uniform mesh with n meshpoints. What is the smallest value of n which is guaranteed to yield a uniform error smaller than 10^{-8}?

Question 6. Derive the Peano kernel for the midpoint rule:

$$\int_0^1 f(x)\,dx - f(1/2) = \int_0^1 K(t) f''(t)\,dt$$

Question 7. Find a linear multistep method of order 3 of the form,

$$y_{n+2} = y_n + h[b_2 f(t_{n+2}, y_{n+2}) + b_1 f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n)],$$

and explain in detail how you derived the unknown coefficients b_0, b_1, b_2. Determine the stability and convergence properties of the method you derived, and explain your reasoning.

Question 8. Consider the following implicit Runge–Kutta method,

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

(a) Determine if the method is a collocation Runge–Kutta method, and explain your reasoning.

(b) Determine the order of accuracy of the method, and explain your reasoning.