COMPLEX ANALYSIS QUALIFYING EXAM
TUESDAY MAY 26TH

General instructions: 3 hours. No notes or textbooks are allowed. The exam has two parts. In part I you are asked to reproduce statements and proofs from the class and to answer true or false questions. In part II you are asked to solve problems similar to ones given in the homework. Please state all results and hypotheses you use and present your solutions clearly, with appropriate detail.

Notation: A **region** $U \subset \mathbb{C}$ is a connected open subset. An **entire function** is is a holomorphic function $\mathbb{C} \longrightarrow \mathbb{C}$.

Part I

1. (40pts) Pick four of the following results from the course and state them clearly (5pts each). Then pick two of those four results and sketch their proofs (10 points each).
Cauchy's Theorem; Cauchy's integral formula; Morera's Theorem; Liouville's Theorem; Casorati-Weierstrass; Open mapping theorem; Maximum principle; Residue Theorem; Argument Principle; Weierstrass' Theorem; Hurwitz's Theorem; Reflection Principle; Riemann Mapping Theorem.

2. (25pts) Are the following statements true or false? If true, give a brief proof. If false, give a counterexample.
(i) If u and v are two harmonic functions on a region U and the set of points where u and v are equal contains an accumulation belonging to U then $u = v$.
(ii) There is a nonconstant holomorphic function $f(z)$ on the region

$$U = \mathbb{C} - \{\, z \in \mathbb{C} \mid \operatorname{Im} z = 0, \operatorname{Re} z < 0 \,\}$$

such that $|f(z)| \leq 1$.
(iii) There are polynomials $p_1, p_2, \dots$ such that

$$\left| \frac{1}{z(z-3)} - p_n(z) \right|$$

converges uniformly to zero on the annulus,

$$U = \{\, z \in \mathbb{C} \mid 1 < |z| < 2 \,\},$$

as n tends to infinity.

(iv) If $f(z)$ is an entire function then either the image of f is dense in $\mathbb{C}$ or $f(z)$ is constant.
(v) Let $\pi\colon \mathcal{O} \longrightarrow \mathbb{C}$ be the sheaf of germs of holomorphic functions over $\mathbb{C}$. Assume that there are germs $\zeta, \zeta_n \in \mathcal{O}$ such that $\lim_{n\to\infty} \zeta_n = \zeta$. Then there is a function element (f, D) such that $\zeta_n = (f)_{\pi(\zeta_n)}$ for n sufficiently large.

Part II

3. (10pts) How many roots does the polynomial

$$z^4 - 6z + 3 = 0,$$

have in the annulus

$$U = \{\, z \in \mathbb{C} \,|\, 1 < |z| < 2 \,\}?$$

4. (10pts) Evaluate

$$\int_0^\infty \frac{\log x}{x^2 - 1}\,\mathrm{d}x.$$

5. (10pts) Show that

$$\pi^2 \frac{\cos \pi z}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n)^2}.$$

(*Hint:* Compare singular parts and use periodicity.)

6. (10pts) Let $f(z)$ be an elliptic function with periods ω_1, ω_2. Assume that $f(z)$ has no zeros or poles on ∂P, where

$$P := \{\, z = s_1\omega_1 + s_2\omega_2 \,|\, 0 < s_1, s_2 < 1 \,\}$$

and let $a_1, a_2, \ldots, a_m$ denote the zeros and $b_1, b_2, \ldots, b_n$ the poles of $f(z)$ in P (each repeated according to multiplicity). State and prove a theorem regarding

$$\sum_{j=1}^{m} a_j - \sum_{k=1}^{n} b_k.$$

Advice: The maximum number of points awarded will depend on the theorem stated...