# Complex Analysis Qualifying Exam – Spring 2022

Name: ______________________________

Student ID: ___________________________

**Instructions:** 3 hours. Open book: Conway and personal notes from lectures may be used. You may use without proof results proved in Conway I-VIII, X-XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

**Notation and terminology:** A region is an open and connected subset of $\mathbb{C}$. The space of analytic (resp., meromorphic) functions in $G$ is denoted by $H(G)$ (resp., $M(G)$).

| Question | Score | Maximum |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| Total | | 60 |

**Problem 1.** [*10 points.*]

Let $G \subset \mathbb{C}$ be a bounded, simply connected region and let $a \in G$. Let $f$ be an analytic self-map of $G$ (i.e., $f(G) \subset G$) such that $f(a) = a$ and $f'(a) = 1$. Show that $f(z) = z$.

**Problem 2.** [*10 points; 4, 4, 2.*]

Let $p(z)$ be a nonconstant polynomial of $z$. Let $G \subset \mathbb{C}$ be a component of the set $\{z\colon |p(z)| < 1\}$.

(a) Show that $p$ has at least one zero in $G$.

(b) Let $f$ be analytic in $G$ with $|f| \leq 1$. Assume that $f$ has a zero at every zero of $p$ such that the order of vanishing of $f$ is at least that of $p$. Show that $|f(z)| \leq |p(z)|$ and if $z = a$ is a zero of $p$ of order $k$, then $|f^{(k)}(a)| \leq |p^{(k)}(a)|$.

(c) If either $|f(a)| = |p(a)|$ for some $z = a$ that is not a zero of $p$ or if $|f^{(k)}(a)| = |p^{(k)}(a)|$ for some $z = a$ that is a zero of $p$ of order $k$, then $f(z) = cp(z)$ for some constant $c$.

**Problem 3.** [*10 points.*]

Consider the function

$$f(z) = \frac{z^2 + 1}{z^2 - 1}$$

in $G = \{z \colon |z| > 2\}$. Does $f$ have a primitive in $G$ (i.e., $F \in H(G)$ such that $F' = f$)? Prove your assertion.

**Problem 4.** [*10 points.*]

Let $G \subset \mathbb{C}$ be a region such that $0 \notin G$ and $G$ is *not simply connected*. Show that the following are equivalent:

(i) $\mathbb{C}_\infty \setminus G$ has precisely two components $F_0$, $F_\infty$ such that $0 \in F_0$, $\infty \in F_\infty$.

(ii) Every $f \in H(G)$ can be approximated in $H(G)$ by rational functions with poles only in $\{0, \infty\}$.

**Problem 5.** [*10 points.*]

Let $G \subset \mathbb{C}$ be an open set, $\{f_n\}$ a sequence in $M(G)$, and $f$ a meromorphic function such that $f_n \to f$ in $M(G)$. Suppose $a \in G$ is a pole of $f$. Show that there is a sequence $\{a_n\}$ in $G$ such that $a_n \to a$ and $f_n$ has a pole at $a_n$ for sufficiently large $n$.

**Problem 6.** [*10 points.*]

Let $h$ be a bounded harmonic function on the unit disc $\mathbb{D} = \{z : |z| < 1\}$. Assume that

$$\limsup_{z \to a} h(z) \leq 0$$

for all $a \in \partial\mathbb{D} \setminus \{1\}$. Show that $h \leq 0$ in $\mathbb{D}$.