Math 220: Spring 2013
Qualifying Exam

Wednesday, May 22	1:00p – 4:00p	AP&M 6402

Instructions: This is a 3-hour exam. No notes or textbooks are allowed. There are two parts to this exam. In Part I, you are asked to reiterate statements and proofs of some results from this course, and to anwer some true and false questions. In Part II, you are asked to solve several problems similar to those you've seen in homework sets and exams in 220ABC. Make sure to state all results and hypotheses used, and present your solutions clearly, with appropriate detail.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ denotes the open unit disk; $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} \colon |z - z_0| < r\}$ denotes an open disk; $\mathbb{A}(z_0; r_1, r_2) = \{z \in \mathbb{C} \colon r_1 < |z - z_0| < r_2\}$ denotes an open annulus; and $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ denotes the upper half-plane.

Part I

1. (*40 points*) Choose 4 of the following theorems covered in this course, and write out their statements carefully and completely (*5 points each*). From among these, choose 2, and sketch their proofs (*10 points each*).

The Cauchy Estimates and Liouville's Theorem
the Open Mapping Theorem
Rouché's Theorem
Hurwitz's Theorem
the Riemann Mapping Theorem
Runge's Theorem
the Mittag-Leffler Theorem
the Poisson Integral Formula
the Maximum Modulus Principle
the Monodromy Theorem
the Great Picard Theorem
Carathéodory's Theorem

2. (*20 points*) Determine if the following statements are True or False. If True, give a brief proof. If False, give a counterexample or prove your assertion otherwise.

(a) (*5 points*) Let $U \subseteq \mathbb{C}$ be an open set, and let $u \colon U \to \mathbb{R}$ be a non-constant harmonic function. Set

$$Z = \{z \in U \colon u_x(z) = u_y(z) = 0\}.$$

It is possible for Z to have an accumulation point in U.

(b) (*5 points*) There is a meromorphic function f on $\mathbb{C} \setminus (2\mathbb{Z})^2$ such that, if $n, m \in 2\mathbb{Z}$ are even integers, then f has a pole of order $n^2 + m^2$ at (n, m), and if $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ with at least one odd, then f has a zero of order $n^2 + m^2$ at (n, m).

(c) (*5 points*) Let $\{u_n\}$ be a sequence of harmonic functions in $\mathbb{D}$. If u_n converges to u uniformly on compact subsets, then u is harmonic on $\mathbb{D}$.

(d) (*5 points*) There is a sequence of holomorphic polynomials p_n such that

$$\lim_{n \to \infty} \sup_{2 \le |z| \le 3} \left| \frac{1}{z^2(z-5)^3} - p_n(z) \right| = 0.$$

Part II

3. (*10 points*) Determine the Laurent series expansions of the function $f\colon \mathbb{C}\setminus\{0,\pm 2\}\to\mathbb{C}$ given by

$$f(z) = \frac{1}{z^3 - 4z}$$

in the annuli $\mathbb{A}(0;0,2)$ and $\mathbb{A}(0;2,\infty)$.

4. (*10 points*) How many zeroes (counted with multiplicity) does the polyonmial

$$f(z) = z^7 - 2z^5 + 6z^3 - z + 1$$

have in the unit disk $\mathbb{D}$?

5. (*10 points*) Let x_n be a sequence of distinct real numbers. Show that there exists a function holomorphic on the upper half-plane, $f \in \mathrm{Hol}(\mathbb{C}_+)$, such that $f(i + x_n) = 0$ for all n *if and only if*

$$\sum_n \frac{1}{x_n^2 + 4} < \infty.$$

6. (*10 points*) Let $U \subset \mathbb{C}$ be the "rounded square" given by the intersection of the four open disks

$$U = \mathbb{D}(1,2) \cap \mathbb{D}(-1,2) \cap \mathbb{D}(i,2) \cap \mathbb{D}(-i,2).$$

Let $S \subset \partial U$ be the curve $S = \{-i + 2e^{it}\colon t \in (\frac{\pi}{4}, \frac{\pi}{2})\}$. Suppose that $f \in \mathrm{Hol}(U)$, and that for any sequence $z_n \in U$ such that $\lim_{n\to\infty} z_n \in S$, it follows that $\lim_{n\to\infty} f(z_n) = 0$. Prove that $f \equiv 0$.