160 points   Complex Analysis Qualifying Examination   May 20, 2009

<u>JUSTIFY EVERYTHING BY CITATION OR PROOF</u>.

(20) 1. Suppose $f(z) = \sum_{n=0}^{N} a_n z^n$ where the $a_n \in \mathbb{C}$ and $N > 0$. Let M be the maximum of $|f(z)|$ on the unit circle about the origin.

i) (5) Show that $|a_0| \le M$.

ii)(15) Show that $|a_N| \le M$.

(20) 2. Find the number of zeros of $f(z) = z^7 + 2z^3 + 4$ in the interior of the first quadrant (all $z = x + iy$ with x and y positive).

(20) 3. Suppose that $\alpha, \beta, \gamma, \delta$ are distinct complex numbers. Show that

$$\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0 .$$

(Hint: this is not an algebra qual.)

(20) 4. Suppose $f(z)$ is analytic at $z = 0$. Prove that there is an integer $n > 0$ such that

$$|f^{(n)}(0)| < n^n \cdot n! \ .$$

(20) 5. Suppose $\{f_n(z)\}_{n \ge 1}$ is a sequence of analytic functions on a reqion A which converges uniformly on A to a function $f(z)$. Show that $f(z)$ is analytic on A and that the sequence of derivatives $\{f_n'(z)\}_{n \ge 1}$ converges uniformly to $f'(z)$ on compact subsets of A.

(20) 6. State and prove the Weierstrass Product Theorem. You may use any general convergence criteria without proof, but you should state what these criteria are.

(20) 7. Evaluate $\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\,dx$ for all real $t \ge 0$.

(20) 8. Find all entire functions $f(z)$ with the property that for all z,

$$|f(z)| \le e^{xy}.$$

(Here, x and y are the real and imaginary parts of z.)