Complex Analysis Qualifying Exam August 28th, 2019

Name (PRINT):________________________________ PID:______________

Signature: ________________________________

Instructions: 3 hours. You may use without proof results proved in Conway up to and including Chapter XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: The unit disk is denoted by $\mathbb{D}$. G is a region, i.e., an open and connected subset of $\mathbb{C}$. $B(a, r)$ denotes the open disk of radius r centered at a. The space of analytic functions in G is denoted by $H(G)$.

Problem	Points	Score
#1	15	
#2	15	
#3	15	
#4	15	
#5	15	
#6	15	
#7	15	
Total	105	

1. Compute the integral $\int_0^\infty \frac{x^2}{x^4+5x^2+4}dx$.

2. Let f be analytic on $\mathbb{D}$ with $|f(z)| \leq \frac{1}{2}$ on $\mathbb{D}$ and $f(0) = r \in \mathbb{R}$. Here $0 < r < \frac{1}{2}$.
(a). Prove that $f(z)$ has no zeros in the disk $\{|z| < 2r\}$.

(b). Can $f(z)$ have a zero on the circle $\{|z| = 2r\}$? If so, find all such functions $f(z)$.

3. Let $A_1 = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $A_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove A_1 and A_2 are not conformally equivalent.

4. Let $b \in \mathbb{D}$ and set $f(z) = z^7 - 2z^5 + b$.

(a). How many roots (counting multiplicity) does f have in $\mathbb{D}$? How many simple roots does f have in $\mathbb{D}$?

(b). How many simple roots does f have in $\{1 \leq |z| < 2\}$?

5. Let $f = \frac{1}{(z-1)(z-5)}$.

(a). Prove that there is a sequence of rational functions $R_n(z)$ whose poles can only occur at 2 and 6 such that

$$\lim_{n\to\infty} \sup_{3\le |z|\le 4} |f(z) - R_n(z)| = 0. \tag{1}$$

(b). Does there exist a sequence of rational functions $R_n(z)$ whose poles can only occur at 6 such that (1) holds? Justify your answer.

6. Find all analytic functions f on $\mathbb{C} \setminus \{0\}$ with the following property:

$$\text{There is some constant } C > 0 \text{ such that } |f(z)| \leq C|z|^2 + \frac{C}{|z|^{\frac{1}{2}}}, \quad \forall z \in \mathbb{C} \setminus \{0\}.$$

7. Let $G = \mathbb{D} \setminus \{0\}$ and f be the function on ∂G such that $f(z) = 0$ for $|z| = 1$ and $f(0) = 1$. Show that the Perron function $u(z)$ of f,

$$u(z) = \sup\{\phi(z) \colon \phi \text{ is subharmonic and } \forall a \in \partial G,\ \limsup_{\zeta \to a} \phi(\zeta) \le f(a)\},$$

is identically zero.

Hint: Consider the family of functions $u_\epsilon(z) = \frac{\log|z|}{\log \epsilon}$ in the annulus $\epsilon < |z| < 1$ for $\epsilon > 0$.