Complex Analysis Qualifying Exam – Fall 2024

Name: ______________________________

Student ID: __________________________

Instructions:

You do not have to reprove any results from Conway or shown in class. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that there exist complex numbers $z \neq 0$ for which $f(z)$ is positive real.

(You may not use Picard's theorem for this question.)

Problem 2. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that for any $a \in \mathbb{R}$, at least one coefficient in the Taylor expansion of f around a is a rational number. Prove that f is a polynomial.

Problem 3. [*10 points.*]

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Assume that

(i) $|f(z)| < 2$ for all $z \in \mathbb{D}$,

(ii) $f\left(\frac{1}{3}\right) = 1$.

Show that f has no zeros in the disc $|z| < \frac{1}{7}$.

Problem 4. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that

(i) f takes real values on the real axis,

(ii) f takes purely imaginary values on the line $\mathrm{Re}\ z = \mathrm{Im}\ z$.

Prove that f takes real values on the imaginary axis.

Problem 5. [*10 points.*]

Let $f_n : \mathbb{D} \to \mathbb{C}$ be holomorphic functions such that

$$\int_{\mathbb{D}} |f_n| \, dx \, dy \leq 1, \quad \forall n \geq 1.$$

Show that there exists a subsequence of $\{f_n\}$ that converges locally uniformly on $\mathbb{D}$.

Problem 6. [*10 points; 5, 5.*]

Let f be meromorphic in $\mathbb{C}$ with finitely many zeros and poles. Write $\alpha_1, \dots, \alpha_n$ for the zeros and poles of f, and let $m_1, \dots, m_n$ be their orders. Assume that

- the poles of f are in the unit disc $\mathbb{D}$,
- $|f(z) - 1| \le \frac{1}{|z|^2}, \quad \forall |z| \ge 1\,.$

(i) Show that f is a rational function.

(ii) Show that $\sum_{i=1}^{n} m_i \alpha_i = 0$.

Problem 7. [*10 points.*]

Let $\mathbb{H} = \{z : \text{Im}z > 0\}$ denote the upper half plane. Let $u : \overline{\mathbb{H}} \to \mathbb{R}$ be a continuous bounded function which is harmonic in $\mathbb{H}$ and $u = 0$ on $\partial\mathbb{H}$. Show that u is constant.