MATH 220

Qualifying Exam, September 7, 2010

Instructions: 3 hours. You may use without proof all results proved in Conway. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: A domain is an open and connected subset of the complex plane $\mathbb{C}$. The unit disk is denoted by $\mathbb{D}$.

1. (50p) Determine if the following statements are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample or prove your assertion otherwise. If you claim your assertion follows from a theorem in Conway, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

(a) (10p) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. If $|a| = R$ and the power series does not converge at $z = a$, then $f(z)$ cannot be analytically continued to an open neighborhood of a.

(b) (10p) Let f and g be analytic functions defined in an open set $G \subset \mathbb{C}$. If for some $a \in G$, $[f]_a = [g]_a$ (where $[f]_a$ and $[g]_a$ denote the germs of f and g at a respectively) then $f(z) = g(z)$ for all $z \in G$.

(c) (10p) Let (X, d) be a metric space, $x \in X$, and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. If every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a subsequence that converges to x, then $\{x_n\}_{n=1}^{\infty}$ converges to x.

(d) (10p) Let G denote the intersection between the disks given by $|z-2| < 3$ and $|z+2| < 3$. For any two points $a, b \in G$, there exists an automorphism of G (i.e. an analytic bijection of G onto itself) sending a to b.

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2. (30p) Find all functions $f(z)$ that satisfy the two requirements:

(i) $f(z)$ is meromorphic in the plane $\mathbb{C}$.

(ii) There exists a constant $C > 0$ such that

$$|f(z) - \tan z| \leq C|f(z)|$$

for all z outside the poles of $f(z)$ and $\tan z$.

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3. (30p) Let G be a domain and $\{f_n\}_{n=1}^{\infty}$ a sequence of analytic functions in G that converge to f in $H(G)$. Assume that each f_n is 1-to-1. Show that f is either constant or 1-to-1.

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4. (30p) Consider the rational function

$$R(z) := \frac{1}{z(z-3i)}.$$

(a) (15p) Prove that there is a sequence of rational functions $R_n(z)$ with poles at 1/3 and 3 such that

$$\lim_{n\to\infty} \sup_{1\le|z|\le 2} \Big|R(z) - R_n(z)\Big| = 0. \tag{1}$$

(b) (15p) Does there exist a sequence of polynomials $R_n(z)$ such that (1) holds? Prove your assertion.

5. (60p) Let $p(z)$ be a nonconstant polynomial and let G be a connected component of the open set $\{z\colon |p(z)| < 1\}$.

(a) (30p) Show that $p(z)$ must have at least one zero in G.

(b) (30p) Let $f(z)$ be analytic in G, satisfying $|f(z)| \le 1$ there. Assume that $f(z)$ vanishes at every zero of $p(z)$ in G and that the vanishing order of $f(z)$ at each such zero is at least that of $p(z)$. Show that:

(i) $|f(z)| \le |p(z)|$ in G.

(ii) If $a \in G$ is a zero of $p(z)$ of order k, then

$$|f^{(k)}(a)| \le |p^{(k)}(a)|. \tag{2}$$

Moreover, if for some such a we have equality in (2), then $f(z) = cp(z)$ for some constant unimodular constant c.