COMPLEX ANALYSIS QUALIFYING EXAM
THURSDAY SEPTEMBER 10TH

General instructions: 3 hours. No notes or textbooks are allowed. The exam has two parts. In part I you are asked to reproduce statements and proofs from the class and to answer true or false questions. In part II you are asked to solve problems similar to ones given in the homework. Please state all results and hypotheses you use and present your solutions clearly, with appropriate detail.

Notation: A **region** $U \subset \mathbb{C}$ is a connected open subset. An **entire function** is a holomorphic function $\mathbb{C} \longrightarrow \mathbb{C}$.

Part I

1. (40pts) Pick four of the following results from the course and state them clearly (5pts each). Then pick two of those four results and sketch their proofs (10 pts each).
Cauchy's Theorem; Cauchy's integral formula; Morera's Theorem; Liouville's Theorem; Casorati-Weierstrass; Open mapping theorem; Residue Theorem; Argument Principle; Weierstrass' Theorem; Hurwitz's Theorem; Reflection Principle; Riemann Mapping Theorem.

2. (25pts) Are the following statements true or false? Give a brief proof in each case.
(i) Let $f(z)$ be an entire function and let $u(z)$ and $v(z)$ be the real and imaginary parts of f, so that $f = u + iv$. There is a non-constant function $f(z)$ such that $u^2 \leq v^2 + 2015$.
(ii) If $f(z)$ is an entire function such that the function $g(z) = f(1/z)$ has a pole at zero then f is surjective.
(iii) There is a conformal map from the region

$U = \{ z = x + iy \,|\, 0 < x < 1 \}$ to the region $\Delta = \{ z \,|\, |z| < 1 \}$.

(iv) There are polynomials $p_1, p_2, \ldots$ such that

$$\left| \frac{1}{z(z-4)} - p_n(z) \right|$$

converges uniformly to zero on the annulus,

$$U = \{ z \in \mathbb{C} \,|\, 2 < |z| < 3 \},$$

as n tends to infinity.

Part II

3. (10pts) Suppose that $a_0, a_1, \ldots, a_n$ is a non-increasing sequence of real numbers, so that $a_0 \geq a_1 \geq a_2 \ldots$. Show that the polynomial

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

has no roots inside the unit disc $|z| < 1$.

4. (10pts) Evaluate

$$\int_0^\infty \frac{\cos x}{x^6 + 1}\, \mathrm{d}x.$$

5. (10pts) If u is harmonic and bounded in $0 < |z| < \rho$ then show that u may be extended to a harmonic function on the whole disk $|z| < \rho$.

6. (10pts) Let $f(z)$ be a meromorphic function. A complex number ω is called a **period** of f if $f(z + \omega) = f(z)$ for all z.
(i) Show that if ω_1 and ω_2 are periods of f then $n_1\omega_1 + n_2\omega_2$ is a period of f for all integers n_1 and n_2.
(ii) Show that there are at most finitely many periods of f in any bounded subset of the complex plane.