Complex Analysis Qualifying Exam May 15th, 2019

Name (PRINT):______________________________ PID:_______________

Signature: ______________________________

Instructions: 3 hours. You may use without proof results proved in Conway up to and including Chapter XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: The unit disk is denoted by $\mathbb{D}$. G is a region, i.e., an open and connected subset of $\mathbb{C}$. $B(a,r)$ denotes the open disk of radius r centered at a. The space of analytic functions in G is denoted by $H(G)$.

Problem	Points	Score
#1	15	
#2	15	
#3	15	
#4	15	
#5	15	
#6	15	
#7	15	
Total	105	

1. Let f be an entire function. Assume $|f| \leq \log(|f|+2)$ on $\mathbb{C}$. Prove f is constant.

2. Let $U \subset \mathbb{C}$ be an open set and f a continuous function on U. Assume f^2 is analytic on U. Prove f is analytic on U.

3. Let f be analytic in $B(0, 1+\epsilon)$ for some small $\epsilon > 0$. Assume $|f(z)| < 1$ for all $|z| = 1$. Prove there is a unique z_0 with $|z_0| < 1$ such that $f(z_0) = z_0$.

4. Let $\Omega \neq \mathbb{C}$ be a simply connected region with $a \in \Omega$ and f a one-to-one analytic function from Ω onto $\mathbb{D}$. Assume $f(a) = 0$ and $f'(a) > 0$. Prove that

$$\inf_{z \in \partial\Omega} |z - a| \leq \frac{1}{f'(a)} \leq \sup_{z \in \partial\Omega} |z - a|.$$

5. Let f be an analytic function in $B(0,2)$. Prove the sequence $g_N(z) := \sum_{n=1}^{N} \frac{f^{(n)}(z)}{n!}$, $N \geq 1$, converges in the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$.

6. Let f be a continuous function on $\{z \in \mathbb{C} : 0 < |z| \leq 1\}$ that is analytic on $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Assume $f(z) = 0$ for every $z = e^{i\theta}$ with $\frac{\pi}{4} < \theta < \frac{\pi}{3}$. Prove $f \equiv 0$.

7. Let f be a bounded, piecewise continuous function on $\partial\mathbb{D}$, and consider the harmonic function

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt, \quad z = re^{i\theta},$$

where $P_r(t) := \mathrm{Re}\frac{1+re^{it}}{1-re^{it}}$ is the Poisson kernel in $\mathbb{D}$. Assume that f is continuous at $a = e^{i\theta_0}$, and show that

$$\lim_{z\to a} u(z) = f(a).$$