Complex Analysis Qualifying Exam – Fall 2017

Name: ____________________________________

Student ID: ________________________________

Instructions:

You have 3 hours. No textbooks and notes are allowed. Solve 7 of the following 8 questions.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		70

Problem 1. [*10 points.*]

Using the calculus of residues, compute

$$\int_0^\infty \frac{\log^2 x}{1+x^2}\, dx.$$

You may wish to use the contour consisting of two half circles in the upper half plane, one of small radius and another of large radius, connected by two line segments along the real axis.

Problem 2. [*10 points.*]

Let $U \subset \mathbb{C}$ be an open set that contains the closed unit disc $\overline{\mathbb{D}} \subset U$. Show that for all holomorphic functions $f : U \to \mathbb{C}$ we have

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \geq 1.$$

Problem 3. [*10 points.*]

Find a bijective analytic function f from $\{z : |z| < 1, \mathrm{Re}\ z > 0\}$ to $\{z : \mathrm{Im}\ z > 0\}$.

Problem 4. [*10 points.*]

Let $\mathfrak{h}^+ = \{z : \text{Im } z > 0\}$ denote the upper-half plane. Let $\mathcal{F}$ be the family of holomorphic functions $f : \mathfrak{h}^+ \to \mathbb{C}$ such that

$$f(i) = 0 \text{ and } |f(z)| < 1 \text{ for all } z \in \mathfrak{h}^+.$$

Find the maximum value of $|f(2i)|$ for $f \in \mathcal{F}$.

Problem 5. [*10 points.*]

Prove that the Riemann surface of the complete analytic function associated to a branch of $\log z$ is simply connected.

Problem 6. [*10 points.*]

(a) Prove that for any distinct real numbers r and ρ,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{it}-\rho|\,dt = \max(\log r, \log|\rho|).$$

(b) Show that the series

$$u(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log\left|z - \frac{1}{2^n}\right|$$

defines a subharmonic function.

Problem 7. [*10 points.*]

Assume that f and g are entire functions without common zeros. Show that there exist entire functions A, B such that

$$Af + Bg = 1.$$

Hint: Ensure that $A = (1 - Bg)/f$ is entire by matching the principal parts of $1/f$ and Bg/f.

Problem 8. [*10 points.*]

Let $n \geq 1$ be an integer. Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that

$$|u(x)| \leq C(1 + |x|^n) \text{ for all } x \in \mathbb{C}.$$

Show that u is a polynomial.