Complex Analysis Qualifying Exam – Spring 2021

Name: ______________________________

Student ID: ______________________________

Instructions:

The exam is closed notes, closed books, no internet, no outside help.

You do not have to reprove any results from Conway or shown in class. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

How many solutions, counted with multiplicities, does the equation

$$z^3 \sin z + 5z^2 + 2 = 0$$

have in the unit disc $|z| < 1$?

Problem 2. [*10 points; 5, 5.*]

Let K be a proper closed arc of the unit circle $|z| = 1$.

(i) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly in K?

(ii) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly on the circle $|z| = 1$?

Please justify your answers.

Problem 3. [*10 points.*]

Let $\mathcal{F}$ denote the family of holomorphic functions $f : \Delta \to \mathbb{C}$ such that

(i) f omits all strictly negative real numbers, and

(ii) $f(0) = 1$.

Find the maximum value of $|f'(0)|$ as $f \in \mathcal{F}$.

Problem 4. [*10 points.*]

Let $f : G \to \mathbb{C}$ be a holomorphic function in $G = \{z : |z| < 2\}$ such that $|f(z)| < 1$ for $z \in G$. Assume that

$$f(1) = f(-1) = f(i) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{15}.$$

Problem 5. [*10 points.*]

Let $a_n = 1 - \frac{1}{n}$ for $n \geq 2$. Show that there are no bounded holomorphic functions $f : \Delta \to \mathbb{C}$ with zeros only at the a_n's.

Problem 6. [*10 points.*]

Let $\{u_n(x, y)\}$ be a sequence of harmonic functions in an open connected set $G \subset \mathbb{R}^2$, converging uniformly on compact subsets of G.

Show that the sequence of partial derivatives $\frac{\partial u_n}{\partial x}$ converges uniformly on compact subsets of G.

Problem 7. [*10 points.*]

Let $\{f_n\}$ be a sequence of automorphisms of the unit disc Δ, converging locally uniformly in Δ to a nonconstant function f. Show that f is an automorphism of Δ.

Hint: Examine the family $\mathcal{F}$ consisting of the inverse automorphisms $f_n^{-1} : \Delta \to \Delta$.