Complex Analysis Qualifying Exam – Fall 2020

Name: ______________________________

Student ID: ___________________________

Instructions:

You do not have to reprove any results from Conway or shown in class. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

Let γ be the closed curve given by the circle $|z - \frac{i}{2}| = 1$ traversed once in the positive (counter-clockwise) direction. Compute

$$\int_\gamma \frac{\sin(i\pi z/2)}{z^2+1} dz.$$

Problem 2. [*10 points; 3, 7.*]

Let $a \in \mathbb{R}$ and $a > 2$. Consider the following equation

$$a + z - e^{2z} = 0. \tag{1}$$

(a) Prove that if $z_0 \in \{\text{Re}(z) < 0\}$ is a solution of the equation (1), then $|z_0 + a| < 1$.

(b) Prove the equation (1) has exactly one solution on the left half plane $\{\text{Re}(z) < 0\}$. Furthermore, prove that this solution must be a real number.

Problem 3. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}, \quad f^{(n)} = \frac{d^n f}{dz^n}$$

converges uniformly on compact subsets of $\mathbb{C}$.

Problem 4. [*10 points.*]

Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $|f(z)| = 2$ everywhere on $\{|z| = 1\}$, and $f^{(3)}(0) = -12$. Here $f^{(3)}$ denotes the third derivative of f.

Problem 5. [*10 points.*]

Let $\mathcal{F}$ be a non-empty family of analytic functions on the unit disc $\mathbb{D}$. Assume for every $f \in \mathcal{F}$, it holds that

$$\int_{\mathbb{D}} |f(z)|(1-|z|)^5 \, dm < 10.$$

Prove $\mathcal{F}$ is a normal family.

Here dm denotes the standard measure in $\mathbb{R}^2$. That is, $dm = dxdy$ for $z = x + iy$.

Problem 6. [*10 points; 3, 7.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z - \sin z$.

(a) Show that f is an odd entire function of order less or equal to 1.

(b) Possibly using (a), show that f can be represented as a product

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

where $\{a_n\}$ is a sequence of non-zero complex numbers with

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

Problem 7. [*10 points.*]

Let $\mathbb{D}$ denote the open unit disc, and let $\mathbb{D}' = \left\{z \in \mathbb{C} : \left|z + \frac{2}{5}\right| < \frac{2}{5}\right\}$ denote the open disc of center $-\frac{2}{5}$ and radius $\frac{2}{5}$. Let $\Omega = \mathbb{D} \setminus \overline{\mathbb{D}'}$.

Find, with justification, an explicit continuous functions $h : \overline{\Omega} \to \mathbb{R}$, harmonic in Ω, and with boundary values $h = 0$ on $\partial\mathbb{D}$ and $h = 1$ on $\partial\mathbb{D}'$.

Hint: You may wish to use a conformal map to change the domain Ω.