Applied Algebra Qualifying Exam: Part II
May 27, 2011

Do as many problems as you can, but you must attempt at least 5 problems where 1 of the problems is from problems 1-3, one of the problems is from problems 4-7 and at least 2 of the problems are from problems 7-9. The point values are relative values for this part of the exam. Your final score will be scaled so that this part of the exam will represent 60% of your point total.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q}$ equal the rationals and $\mathbb{C}$ denote the complex numbers.
If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of n, let A^λ denote the irreducible representation of the symmetric group S_n such that the Frobenius image of $\chi^{A^\lambda} = \chi^\lambda$ is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$.

1) (30 pts) Let G be a finite group and $A : G \to GL(n, \mathbb{C})$ be a representation of G.

(a) Show that if the only matrices S which commute with $A(g)$ for all $g \in G$ are of the form λI, then A is irreducible.

(b) Show that if G is abelian, then there is an invertible matrix T such that for all $g \in G$, $TA(g)T^{-1}$ is a diagonal matrix.

(c) Show that if $g \in G$ is in the center of G, then $A(g) = cI_n$ for some nonzero constant $c \in \mathbb{C}$.

2) (30 pts.) (d) Prove that if $\lambda(x)$ is a linear character of a finite group G, then for any irreducible character χ of G, the function χ^* defined by $\chi^*(\sigma) = \lambda(\sigma)\chi(\sigma)$ for all $\sigma \in G$ is also an irreducible character of G.

(b) Given a partition λ of n, let $\ell(\lambda)$ denote the number of parts of λ and λ' denote its conjugate partition. Let χ^λ_μ denote the value of the character of the irreducible representation A^λ of S_n at the conjugacy class indexed by the partition μ. Show that $\chi^{\lambda'}_\mu = (-1)^{n-\ell(\mu)}\chi^\lambda_\mu$.

3) (40 pts.) Let $G = \{g_1, \ldots, g_k\}$ be a finite group. Introduce variables $x_{g_1}, \ldots, x_{g_k}$ and consider the $k \times k$ matrix

$$X = [x_{g_i g_j^{-1}}].$$

Let $X = \sum_{i=1}^k A(g_i)x_{g_i}$ so that we can define a map $g_i \to A(g_i)$.

(a) Show that A is the left regular representation of G.

(b) Show that $det(X) = \prod_{\nu=1}^{h} det(\sum_{g \in G} A^{(\nu)}(g)x_g)^{n_\nu}$ where $A^{(1)}, \ldots, A^{(h)}$ are a complete set of representatives of the irreducible representations of G and $n_\nu = dim(A^{(\nu)})$ for $\nu = 1, \ldots, h$.

c) Use part (b) to show that

$$det \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{bmatrix} = \prod_{r=0}^{n-1} (x_0 + \epsilon^r x_1 + \epsilon^{2r} x_2 + \ldots \epsilon^{(n-1)r} x_{n-1})$$

where $\epsilon = e^{2\pi i/n}$.

4) (40 pts.)
(a) Use the Murnaghan-Nakayama rule to compute the value of the irreducible characters of S_5 at the conjugacy class indexed by the partition $(2,3)$.

(b) Find the character table for $S_3 \times S_2$ where $S_3 \times S_2$ is the Young subgroup of S_5 consisting of all permuations $\sigma \in S_5$ such that

$$\sigma(1), \sigma(2), \sigma(3) \in \{1,2,3\}, \sigma(4), \sigma(5) \in \{4,5\}.$$

(c) Find the values of the character of $A^{(1,4)}$ on the conjugacy classes of S_5.

(d) Decompose $A^{(1,4)} \downarrow_{S_3 \times S_2}^{S_5}$ as a sum of irreducible representations of $S_3 \times S_2$.

5) (40 pts.)

(a) Let $A^{(1,1,2)} \times A^{(1,2)}$ denote the representation of $S_4 \times S_3$ such that for all $(\sigma, \tau) \in S_4 \times S_3$

$$A^{(1,1,2)} \times A^{(1,2)}(\sigma, \tau) = A^{(1,1,2)}(\sigma) \otimes A^{(1,2)}(\tau)$$

where for any matrices A and B, $A \otimes B$ denotes the tensor product of A and B. Decompose $A^{(1,1,2)} \times A^{(1,2)} \uparrow_{S_4 \times S_3}^{S_8}$ as a sum of irreducible representations of S_7.

(b) Show that $\{A^\lambda \times A^\mu : \lambda \vdash 4 \text{ and } \mu \vdash 3\}$ is a complete set of representatives of the irreducible representations of $S_4 \times S_4$ where $A^\lambda \times A^\mu(\sigma, \tau) = A^\lambda(\sigma) \otimes A^\mu(\tau)$.

Note: For parts (a) and (b) above, regard $S_4 \times S_3$ as a subgroup of S_7 by letting

$$S_4 \times S_4 = \{\sigma \in S_7 : \sigma(1), \sigma(2), \sigma(3), \sigma(4) \in \{1,2,3,4\}, \sigma(5), \sigma(6), \sigma(7) \in \{5,6,7\}\}.$$

(c) Let T denote the trivial representation. Decompose $T \uparrow_{S_1 \times S_3 \times S_3}^{S_7}$ as a sum of irreducible representations of S_7 where $S_1 \times S_3 \times S_3$ is the Young subgroup of S_7 consisting of all permutations $\sigma \in S_7$ such that

$$\sigma(1) = 1, \sigma(2), \sigma(3), \sigma(4) \in \{2,3,4\}, \sigma(5), \sigma(6), \sigma(7) \in \{5,6,7\}.$$

(d) Decompose $A^{(2,2)} \otimes A^{(2,2)}$ as a sum of irreducible representations of S_4 where $\otimes$ represents the Kronecker product of the representations.

6) (40 pts.) Let S_4 denote the symmetric group on 4 elements and A_4 denote the alternating group, i.e. $A_4 = \{\sigma \in S_4 : sign(\sigma) = 1\}$.

(a) Find the conjugacy classes of A_4.

(b) Let $D = \{\epsilon, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Show that D is a normal subgroup of A_4 and that A_4/D is isomorphic to Z_3.

(c) Give the character table for Z_3.

(d) Find the lifting of the irreducible characters of Z_3 to A_4.

(e) Use (d) to complete the character table of A_4.

(7) (30 pts.)
(a) Two ideals I and J in $\mathbb{C}[x_1, \ldots, x_n]$ are co-maximal if and only if $I + J = \mathbb{C}[x_1, \ldots, x_n]$.
(i) Prove that I and J are co-maximal if and only if $V(I) \cap V(J) = \emptyset$.

(ii) Show that if I and J are comaximal, then $IJ = I \cap J$.

(b) Suppose that I and J are ideals in $k[x_1, \ldots, x_n]$ where k is field. Show that if $I \subseteq \sqrt{J}$, then there is an $m \geq 1$ such that $I^m \subseteq J$. (Hint: Use the Hilbert Basis Theorem.)

(8) (40 pts.)

Consider the equations

$$\begin{aligned} xy + x^2 &= 1 \\ y^2 - 2x^2 &= -2 \end{aligned}$$

(a) Let I be the ideal of $\mathbb{C}[x, y]$ generated by these equations. Find the Groebner basis for I relative to lexicographic order where $y > x$.

(b) Find a Groebner basis for $\mathbb{C}[x] \cap I$.

(c) Find all solutions to these equations that lie $\mathbb{C}^2$.

(d) Find a vector space basis for $\mathbb{C}[x, y]/I$.

(9) (40 pts.)
(a) Show that if I is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $V(I)$ is finite, then $\mathbb{C}[x_1, \ldots, x_n]/I$ is finite dimensional when considered as a vector space of $\mathbb{C}$.

(b) Find a reduced Groebner basis for $I = \langle x^2 + xy + y, xy + y^2 \rangle$ with respect to the graded lexicographic order where $x > y$.

(c) Show that $x^2 + y^2 \in \sqrt{I} - I$.