QUALIFYING EXAM: APPLIED ALGEBRA Spring 2007

Try to do as many problems as possible, preferrably with at least two problems from each group. The grading will be rescaled so that it will make up 60% of the complete score of the exam.

Part I: Representation Theory

1. (a) Let H be generated by a and b with the relations $a^3 = 1 = b^3$ and $ab = ba$. i.e. $H \cong \mathbf{Z}/3 \times \mathbf{Z}/3$. Write down all irreducible representations of H.

   Let now G be the group generated by a, b, c, with relations as in (a) as well as with $c^3 = 1$ and $ca^ib^jc^{-1} = a^ib^{i+j}$. You can assume that G has order 27.

   (b) Calculate the matrices of the generators a, b and c in the representation W induced by a given irreducible representation V of H. Determine for which V all elements of H act via multiples of the identity matrix on W. Show directly that in this case the induced representation is reducible.
   (c) Determine all irreducible representations of G.

2. Let B be the matrix given by $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 27 & 0 & 3 \end{bmatrix}$, and let $V = \mathbf{C}^3$ be an irreducible G-module for a finite group G, with $g \in G$ acting via the matrix A_g on V. Let $C = \frac{1}{|G|}\sum_g A_g B A_g^{-1}$.
   (a) Calculate $Tr(C)$, where Tr is the usual trace.
   (b) Calculate C.

3. Let G be a finite group, and let $p = \sum \alpha_g g$ be a minimal idempotent in the simple component of $\mathbf{C}G$ labeled by λ. Let d_λ be the dimension of a simple G-module on which p_λ acts nonzero.
   (a) What is the dimension of the space $p\mathbf{C}S_n$? What is $\chi_{reg}(p)$, where χ_{reg} is the character of the left-regular representation.
   (b) Calculate the coefficient α_1.

II. Symmetric Groups and Symmetric Functions

4. (a) Expand the product of Schur functions $s_\lambda s_{[1^2]}$ into a linear combination of Schur functions for all Young diagrams λ with three boxes.
   (b) Decompose the simple S_5-module $V_{[3,2]}$ into a direct sum of simple $S_3 \times S_2$-modules, with S_3 permuting the letters $\{1, 2, 3\}$ and S_2 permuting the letters $\{4, 5\}$.
   (c) Determine the decomposition of $V_{[3,2]}$ as a direct sum of simple S_3-modules and give the structure (i.e. decomposition into simple matrix algebras) of the commutant of the S_3 action on $V_{[3,2]}$.

5. Let W be the S_9-module obtained by inducing up from the $S_6 \times S_3$ module $V_{[3,3]} \otimes V_{[2,1]}$; here V_λ denotes the simple S_n-module labeled by the Young diagram λ. Determine the decomposition of W into a direct sum of irreducible S_9-modules.

6. Let π be an $(n-1)$-cycle in S_n. Determine all Young diagrams λ with n boxes for which $\chi_\lambda(\pi) \neq 0$. Partial credit if you calculate all characters of an $(n-1)$-cycle for $n = 3, 4$.

Part III: Commutative Algebra and Gröbner Bases

7. Consider the ideal I generated by the polynomials $x^3 - y^2 + 1, x^3 + y^2 + z^2 - 1$ and $y^2 + yz - 1$.
(a) Calculate a Gröbner basis for the lexicographical order $x > y > z$.
(b) Calculate all the solutions given by the common zeros of the generating polynomials, i.e. calculate the variety $V(I)$ of I.

8. Let $G = \mathbf{Z}/3$ act on a two dimensional vector space as a diagonal matrix with diagonal entries being $\theta^{\pm 1}$, where $\theta = e^{2\pi i/3}$.
(a) Find a system of generators for $k[x, y]^G$.
(b) Calculate the Hilbert series for $k[x, y]^G$.
(c) Find at least one relation among the generators in (a). Give a precise description how you would find all possible relations (you need not carry out the calculations).

9. (a) Show the following: Let G be a finite group, and let $g \mapsto A_g \in Gl(\mathbf{C}^n)$ and $g \mapsto B_g \in Gl(\mathbf{C}^n)$ be two equivalent representations of G. Show that the rings of invariants $k[x_1, x_2, \ldots, x_n]^{G(A)}$ and $k[x_1, x_2, \ldots, x_n]^{G(B)}$ defined by these two actions have the same Hilbert series.
(b) Give two examples of an action of a group G on a two dimensional vector space which lead to two different Hilbert series of the corresponding rings $k[x, y]^G$.