# Algebra/Applied Algebra Qualifying Exam
## Part 1
## September 10, 2004

(15) 1. State and prove the Cayley-Hamilton Theorem. (You may use the Schur Decomposition Theorem.)

(10) 2. (a) Show that $a_1, \cdots, a_n \in \mathbb{R}^m$ are linearly independent over $\mathbb{C}$ iff they are linearly independent over $\mathbb{R}$.

(b) Show that if $A \in M_n(\mathbb{R})$, then an eigenvalue $\lambda$ of A is real iff it has a real corresponding eigenvector.

(15) 3. Let $\hat{x}$ be a least squares solution to $Ax = b$, where $A \in M_{m,n}$ and $m \geq n$. Let $A^\dagger$ be the pseudo-inverse of $A$. Use the Singular Value Decomposition to show that $\tilde{x} = A^\dagger b$ is the min 2- norm least squares solution to $Ax = b$, i.e., show

(a) $\tilde{x}$ is a least squares solution,

(b) if $\hat{x}$ is a least square solution then $\|\hat{x}\|_2 \geq \|\tilde{x}\|_2$, and

(c) $\tilde{x}$ is unique.

Notation: $M_{m,n} \equiv$ set of $m \times n$ complex matrices.

$M_n \equiv$ set of $n \times n$ complex matrices.

$M_n(\mathbb{R}) \equiv$ set of $n \times n$ real matrices.

Applied Algebra Qualifying Exam: Part III
Fall 2004, September 10, 2004

Do as many problems as you can, but you must attempt at least 1 problem from problems 1-3, one problem from 4-7 and at least two problems from 7-9. The point values are relative values for this part of the exam. Your final score will be scaled so that this part of the exam will represent 60% of your point total.

Let $N = \{0, 1, 2, \ldots\}$, $Z = \{0, \pm 1, \pm 2, \ldots\}$, $Q$ equal the rationals and $\mathbf{C}$ denote the complex numbers.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of $n$, let $A^\lambda$ denote the irreducible representation of the symmetric group $S_n$ such that the Frobenius image of $\chi^{A^\lambda} = \chi^\lambda$ is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$.

1) (20 pts.) (a) Prove that if $G$ is finite group and $\lambda(x)$ is a linear character of $G$, then for any irreducible character $\chi$ of $G$, the function $\chi^*$ defined by $\chi^*(\sigma) = \lambda(\sigma)\chi(\sigma)$ for all $\sigma \in G$ is also an irreducible character of $G$.

(b) Let $A : G \rightarrow GL_n(\mathbf{C})$ and $B : G \rightarrow GL_n(\mathbf{C})$ be two representations of a finite group $G$. Show that if for all $\sigma \in G$, there exists a matrix $P(\sigma)$ such that

$$(P(\sigma))^{-1} A(\sigma) P(\sigma) = B(\sigma),$$

then there exist a nonsingular matrix $T$ such that for all $\sigma$,

$$T^{-1} A(\sigma) T = B(\sigma).$$

2) (40 pts.) Let $G = \{g_1, \ldots, g_k\}$ be a finite group. Introduce variables $x_{g_1}, \ldots, x_{g_k}$ and consider the $k \times k$ matrix

$$X = [x_{g_i g_j^{-1}}].$$

Let $X = \sum_{i=1}^k A(g_i) x_{g_i}$ so that we can define a map $g_i \rightarrow A(g_i)$.

(a) Show that $A$ is the left regular representation of $G$.

(b) Show that

$$det(X) = \prod_{\nu=1}^{h} det(\sum_{g \in G} A^{(\nu)}(g) x_g)^{n_\nu}$$

where $A^{(1)}, \ldots, A^{(h)}$ are a complete set of representatives of the irreducible representations of $G$ and $n_\nu = dim(A^{(\nu)})$ for $\nu = 1, \ldots, h$.

c) Use part (b) to show that

$$det \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{bmatrix} = \prod_{r=0}^{n-1} (x_0 + \epsilon^r x_1 + \epsilon^{2r} x_2 + \ldots \epsilon^{(n-1)r} x_{n-1})$$

where $\epsilon = e^{2\pi i/n}$.

3) (20 pts.) Given a partition $\lambda$ of $n$, let $l(\lambda)$ denote the number of parts of $\lambda$ and $\lambda'$ denote its conjugate partition. Let $\chi_\mu^\lambda$ denote the value of the character of the irreducible representation $A^\lambda$ of $S_n$ at the conjugacy class indexed by the partition $\mu$. Show that $\chi_\mu^{\lambda'} = (-1)^{n-l(\mu)} \chi_\mu^\lambda$.