Name: ______________________________ S.I.D.: ______________________________

Qualifier Exam in Applied Algebra

September 5, 2017

	Full	Real
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Notes: 1) For computational questions, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof question, no credit will be given for no reasons or wrong reasons.

1. (25 points) Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $\operatorname{rank} A^2 \leq \operatorname{rank} A^3$. Show that

$$\operatorname{rank} A^k \leq \operatorname{rank} A^{k+1}$$

for all integers $k \geq 3$.

2. (25 points) Let $A \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. If

$$\|A\|_F^2 \leq |\lambda_1|^2 + \cdots + |\lambda_n|^2,$$

show that A is normal, i.e., $AA^* = A^*A$.

3. (25 points) Let $A, B \in \mathbb{R}^{n\times n}$ be two real symmetric positive definite matrices. If $A - B$ is positive definite, is $B^{-1} - A^{-1}$ also positive definite? If yes, give a proof; if not, give a counterexample.

4. (5 + 20 points) Let G be a finite group and let $X : G \to GL_3(\mathbb{C})$ be an irreducible 3-dimensional complex matrix representation of G. Let A be the matrix

$$A = \begin{pmatrix} 1 & -12 & 4 \\ 0 & 5 & 3 \\ -2 & 1 & 3 \end{pmatrix}$$

and let $B = \frac{1}{|G|} \sum_{g \in G} X(g) A X(g)^{-1}$.

(a) Determine the trace of the matrix B.

(b) Determine the matrix B.

5. (25 points) Let $D_6 = \langle r, s \mid r^6 = s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of symmetries of a regular pentagon. Calculate the character table of D_6.

6. (8 + 8 + 9 points) For a partition $\lambda \vdash n$, let S^λ be the corresponding irreducible representation of the symmetric group S_n over $\mathbb{C}$.

   (a) Calculate the decomposition of the induced module

   $$V = (S^{(2)} \otimes S^{(2)} \otimes S^{(1)}) \uparrow_{S_2 \times S_2 \times S_1}^{S_5}$$

   into irreducible S_5-modules.

   (b) What is the dimension over $\mathbb{C}$ of the endomorphism algebra $\mathrm{End}_{S_5}(V)$?

   (c) For any permutation $\sigma \in S_5$, let $\mathrm{sign}(\sigma)$ be the sign of σ. Define a linear operator $\varphi : V \to V$ by

   $$\varphi(v) = \frac{1}{5!} \sum_{\sigma \in S_5} \mathrm{sign}(\sigma) v.$$

   What is the rank of the operator φ?

7. (15 + 10 points) Let $I \subseteq \mathbb{C}[x, y]$ be an ideal with vanishing locus

$$\mathbf{V}(I) = \{(2, 3)\} \subset \mathbb{C}^2.$$

(a) Prove that the quotient $\mathbb{C}[x, y]/I$ is finite-dimensional as a $\mathbb{C}$-vector space.

(b) Is the result of the last part still true if we replace $\mathbb{C}$ by $\mathbb{R}$?

8. (25 points) Prove or give a counterexample: Let G and H be two subgroups of the matrix group $GL_2(\mathbb{C})$ which satisfy $G \cong H$ (group isomorphism). Then the invariant rings $\mathbb{C}[x,y]^G$ and $\mathbb{C}[x,y]^H$ have the same Hilbert series.