APPLIED ALGEBRA QUALIFIER

PART I

REPRESENTATION THEORY

100 points gets you full credit more would be heavenly

R.1   Show that a group of order 121 is Abelian.   (10)

R.2   Show that if f is a nilpotent element of the group algebra of a finite group Γ then $\hat{f} = 0$.

Hint: Use the Fourier transform.   (20)

R.3   Let the group Γ act on a family $\mathcal{F}$ and let χ be the character of the representation resulting from this action.

    a) Show that the multiplicity of the trivial in this representation is equal to the number of orbits of $\mathcal{F}$ under the action of Γ.   (20)

    b) Show that the integer $\langle \chi, \chi \rangle_\Gamma$ counts the number of orbits in the action of Γ on the family of ordered pairs   (20)

$$\mathcal{F} \times \mathcal{F} = \left\{ (f, g) : f, g \in \mathcal{F} \right\}$$

    c) Suppose that Γ acts transitively on $\mathcal{F}$. Let f_o be an element of $\mathcal{F}$, H be its stabilizer and let

$$\Gamma = H\tau_1 H + H\tau_2 H + \cdots + H\tau_k H$$

be the double coset decomposition of Γ resulting from the equivalence relation

$$\gamma_1 \sim_H \gamma_2 \iff \gamma_2 = h'\gamma_1 h'' \quad \text{(for some } h', h'' \in H \text{)}$$

Show that in this case $\langle \chi, \chi \rangle_\Gamma = k$.   (30)

Hint: Use part b)

R.4   Give the expansion of the Garnir polynomial $G_T(x)$ corresponding to the tableau

$$T = \begin{array}{ccc} 7 & & \\ 2 & 3 & 5 \\ 1 & 4 & 6 \end{array}$$

in terms of the Garnir polynomials of the standard tableaux of same shape.   (20)

R.5 Let $P(T)$ be the Young idempotent corresponding to the row group of the tableau

$$T = \begin{matrix} 5 & & \\ 2 & 6 & 7 \\ 1 & 3 & 4 \end{matrix}$$

Let χ be the character of the representation resulting from the action of S_7 on the left ideal $\mathcal{A}(S_n)P(T)$. Give the expansion of χ in terms of the character basis $\{\chi^\lambda\}_{\lambda \vdash 7}$ (30)

Hint: Use the Frobenius map.

PART II
SYMMETRIC FUNCTION THEORY

100 points gets you full credit more would be heavenly

ST.1 Use the SF package to compute the two scalar products   (10)

$$a) \quad \frac{1}{5!} \sum_{\sigma \in S_5} \chi^{(2,2,1)}(\sigma)\, \chi^{(3,2)}(\sigma)\, \chi^{(3,1,1)}(\sigma) , \qquad b) \quad \frac{1}{5!} \sum_{\sigma \in S_5} \left(\chi^{(3,2)}(\sigma)\right)^2 \chi^{(3,1,1)}(\sigma)$$

1) Give a representation theoretical interpretation of the resulting integers.   (15)

2) Could you have predicted the answer for b) given the answer for a)?   (20)

ST.2 Show that

$$e^{|c_1|} = \sum_\lambda f^{|\lambda|} \frac{s_\lambda}{h_\lambda}$$ (20)

where h_λ denotes the product of the hooks of λ.

ST.3 Give a representation theoretical interpretation of the identity

$$\partial_{p_1} c_1^n = n c_1^{n-1}$$ (20)

ST.4 Compute the the Schur function expansion of $s_{22} \times s_{2,1}$ by constructing the standard tableaux yielded by the Littlewood Richardson rule.   (20)

a) Check your result by means of the SF package.

b) Give a representation theoretical interpretation of the coefficients in this expansion.   (20)

ST.5 Let Γ be the group of rotations of the cube.

1) Compute the Polya enumerator of the action of Γ on the edges of the cube.   (20)

2) Use the previous result to compute the number of ways to colour the edges of the cube using seven times RED and five times Yellow.   (10)

3) Show that the polynomial you computed in 1) is the Frobenius image of the character of a representation of S_{12}.   (10)

PART III

GRÖBNER BASES METHODS and INVARIANT THEORY

135 points gets you full credit more would be heavenly

GB.1 Use the partial fracion package of Guoce Xin to compute the generating function

$$F_{S_2}(x_1, x_2, x_3, x_4) = \sum_{p \in S_2} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}$$

where the sum is over the compositions $p = (p_1, p_2, p_3, p_4)$ which are solutions of the Diophantine system

$$S_2 = \left\| \begin{matrix} p_1 + p_2 - p_3 - p_4 = 0 \\ p_1 - p_2 + p_3 - p_4 = 0 \end{matrix} \right. \tag{20}$$

GB.2 Use Gröbner bases to find all solutions of the system of equations

$$S_3 = \left\| \begin{matrix} -x^2 + y + z = 0 \\ x - y^2 + z = 0 \\ x + y - z^2 = 0 \end{matrix} \right. \tag{20}$$

GB.3 Construct the group G of 3×3 matrices by applying the Young natural representation indexed by

[2, 1, 1] to the permutations of S_4.   (10)

(1) Compute the Hilbert series $F_{R^G}(q)$ of the ring of G-invariants   (5)

(2) Rewrite it in the form

$$F_{R^G}(q) = \frac{1 + q^d}{(1 - q^{d_1})(1 - q^{d_2})(1 - q^{d_3})} \tag{20}$$

(3) Calculate the first 10 terms of this series.   (5)

(4) Construct three homogeneous G invariants of degrees d_1, d_2, d_3 (20)

(5) Compute the Gröbner basis of the ideal (I_1, I_2, I_3) (5)

(6) Verify that I_1, I_2, I_3 are a system of parameters by checking that the quotient $Q[x_1, x_2, x_3]/(I_1, I_2, I_3)$ has finite dimension. If not go back to step (4).   (5)

(7) Construct a G invariant η of degree d.   (10)

(9) Verify that it is not a polynomial in I_1, I_2, I_3 by computing the polynomial Q such that $Q(\eta, I_1, I_2, I_3) = 0$.   (10)

(10) Based on your previous results show that step (9) is not needed.   (20)

(11) Compute the Jacobian of I_1, I_2, I_3 and construct its linear factors.   (5)

(12) Assuming that there is a reflection group G' that also leaves I_1, I_2, I_3 invariant you could have predicted the number of these linear factors. Why?.   (20)