Name: ______________________ S.I.D.: ______________________

Qualifier Exam in Applied Algebra

May 24, 2018

	Full	Real
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Notes: 1) For computational questions, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof question, no credit will be given for no reasons or wrong reasons.

1. (25 points) Let $A, B \in \mathbb{C}^{8\times 8}$ be two matrices such that

$$\operatorname{rank} A = \operatorname{rank} B = 6, \quad \operatorname{rank} A^2 = \operatorname{rank} B^2 = 4, \quad \operatorname{rank} A^3 = \operatorname{rank} B^3 = 2, \quad \operatorname{rank} A^4 = \operatorname{rank} B^4 = 0.$$

Determine whether A and B are similar to each other or not. If yes, explain why; if no, give a counterexample.

2. (25 points) Let $A \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. Show that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq \sigma_1^2 + \cdots + \sigma_n^2.$$

If the above is an equality, is the matrix A normal (i.e., $AA^* = A^*A$)? If yes, explain why; if no, give a counterexample.

3. (25 points) Let $P \in \mathbb{C}^{n\times n}$ be a positive definite Hermitian matrix. Show that

$$|x^*y|^2 \leq (x^*Px)(y^*P^{-1}y)$$

for all $x, y \in \mathbb{C}^n$.

4. (5 + 20 points) Let X be the set of the 20 diagonals in a regular octogon. The dihedral group D_8 of octogon symmetries acts on X; let $V = \mathbb{C}[X]$ be the associated permutation representation of D_8.

   (a) How many orbits are there in the action of D_8 on X?

   (b) The *Reynolds operator* $R_{D_8} : V \to V$ is given by

   $$R_{D_8}(v) = \frac{1}{|G|} \sum_{g \in D_8} g.v.$$

   What is the rank of R_{D_8}?

5. (25 points) Let A_4 be the index two subgroup of S_4 consisting of even permutations. Find the character table of A_4.

6. (15+5+5 points) For a partition $\lambda \vdash n$, let S^λ be the corresponding irreducible representation of the symmetric group S_n over $\mathbb{C}$.

   (a) Calculate the decomposition of the induced module

   $$V = (S^{(1,1)} \otimes S^{(2)} \otimes S^{(1)}) \uparrow_{S_2 \times S_2 \times S_1}^{S_5}$$

   into irreducible S_5-modules.

   (b) What is the dimension of the endomorphism ring $\mathrm{End}_{S_5}(V)$ as a $\mathbb{C}$-vector space?

   (c) What is the dimension of the *center* of the endomorphism ring $\mathrm{End}_{S_5}(V)$ as a $\mathbb{C}$-vector space?

7. (10+15 points) Let k be a field and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Fix a monomial order $<$ and let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis for I with respect to $<$.

   (a) Explain why the set of cosets

$$\{m + I \,:\, m \text{ a monomial in } x_1, \ldots, x_n \text{ and } \mathrm{LM}(g_i) \nmid m \text{ for } 1 \leq i \leq s\}$$

   is linearly independent in the quotient $k[x_1, \ldots, x_n]/I$.

   (b) Is the conclusion of (a) still true of G is a basis for I which is not necessarily Gröbner? Prove or give a counterexample.

8. (15+10 points) Let $G \subseteq GL_2(\mathbb{R})$ be the order 4 matrix group generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) What is the Hilbert series of the invariant ring $\mathbb{R}[x,y]^G$?

(b) Do there exist algebraically independent homogeneous polynomials $f_1, f_2 \in \mathbb{R}[x,y]^G$ such that $\mathbb{R}[x,y]^G = \mathbb{R}[f_1, f_2]$? Why or why not?