Name: ______________________________ S.I.D.: ______________________________

Qualifier Exam in Applied Algebra

September 5, 2018

	Full	Real
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Notes: 1) For computational questions, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof question, no credit will be given for no reasons or wrong reasons.

1. (25 points) For the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

determine its Jordan's canonical form (JCF) and find a nonsingular matrix P such that $P^{-1}AP$ gives the JCF.

2. (25 points) Let $A \in \mathbb{C}^{n\times n}$ be a matrix such that $|\lambda_i| < 1$ for all its eigenvalues λ_i. Let $B_k = A^k$. Do we necessarily have

$$\lim_{k\to\infty} \left(\sum_{i,j=1}^{n} |(B_k)_{ij}|^3 \right) = 0?$$

If yes, give a proof; if no, give a counterexample.

3. (25 points) Let $A, B \in \mathbb{R}^{n \times n}$ be two symmetric positive definite matrices. If $A - B$ is positive definite, is $B^{-1} - A^{-1}$ necessarily positive definite? If yes, give a proof; if no, give a counterexample.

4. (10 + 15 points) Let D_n be the dihedral group of symmetries of a regular n-gon, let C_n be the cyclic group of order n, and let S_n be the symmetric group on n letters.

   (a) Explain why we have the group isomorphism $D_6 \cong D_3 \times C_2$.

   (b) Write down the character table of D_6.

5. (25 points) Let G be a finite group, let V be a finite-dimensional complex representation of G, and let $\chi : G \to \mathbb{C}$ be the character of V. Let $g \in G$ be a group element such that g is conjugate to g^{-1}. Prove that $\chi(g)$ is a real number.

6. (15 + 10 points) For a partition $\lambda \vdash n$, let S^λ be the corresponding irreducible S_n-module. Let $H = S_3 \times S_3 \times S_1$, so that H is a subgroup of S_7. Let V be the induced representation

$$V = (S^{(3)} \otimes S^{(1,1,1)} \otimes S^{(1)}) \uparrow_H^{S_7} .$$

(a) Find the decomposition of V into irreducible S_7-modules.

(b) What is the dimension of the endomorphism ring $\mathrm{End}_{S_7}(V)$?

7. (10+15 points) Let k be a field and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Fix a monomial order $<$ and let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis for I with respect to $<$.

(a) Explain why the collection of cosets

$$\{m + I \,:\, m \text{ a monomial in } x_1, \ldots, x_n \text{ and } \mathrm{LM}(g_i) \nmid m \text{ for } 1 \leq i \leq s\}$$

is linearly independent in the quotient $k[x_1, \ldots, x_n]/I$.

(b) Is the conclusion of (a) still true if G is a basis for I which is not necessarily Gröbner? Prove or give a counterexample.

8. (15 + 10 points) (a) Let $I \subseteq \mathbb{C}[x, y]$ be an ideal such that $\mathbf{V}(I) = \{(0, 0), (1, 1)\} \subset \mathbb{C}^2$. Prove that the quotient ring $\mathbb{C}[x, y]/I$ is a finite-dimensional $\mathbb{C}$-vector space.

(b) Is the conclusion to (a) still true if we replace $\mathbb{C}$ by $\mathbb{R}$? Justify your answer.