Name: ________________________________ S.I.D.: ________________________________

Qualifing Exam in Applied Algebra

May 30, 2017

	Full	Real
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Notes: 1) For computational questions, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof question, no credit will be given for no reasons or wrong reasons.

1. (25 points) Let $A \in \mathbb{C}^{10\times 10}$ be a matrix such that

$$\operatorname{rank} A = 7, \quad \operatorname{rank} A^2 = 4, \quad \operatorname{rank} A^3 = 1, \quad \operatorname{rank} A^4 = 0.$$

Determine all possibilities of Jordan's canonical form for A.

2. (25 points) Let $A, B \in \mathbb{R}^{n\times n}$ be two real symmetric matrices. If $AB = BA$, show that there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q^T AQ, Q^T BQ$ are both diagonal.

3. (25 points) Let $A, B \in \mathbb{R}^{n \times n}$ be two real matrices. Denote by $\sigma_i(A)$ (resp., $\sigma_i(B)$) the i-th largest singular value of A (resp., B). If $\|Ax\|_2 > \|Bx\|_2$ for all $x \neq 0$, show that $\sigma_i(A) > \sigma_i(B)$ for all $i = 1, \ldots, n$.

4. (25 points) Let Q_8 denote the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the usual multiplication

$$(-1)^2 = 1,\ (-1)i = -i = i(-1),\ (-1)j = -j = j(-1),\ (-1)k = -k = k(-1),$$

$$i^2 = j^2 = k^2 = -1,\ ij = k = -ji,\ ki = j = -ik,\ jk = i = -kj.$$

Calculate the character table of Q_8.

5. (25 points) Consider the action of the symmetric group $\mathfrak{S}_4$ on the vector space $V = \mathbb{C}[x_1, x_2, x_3, x_4]_3$ of homogeneous cubic polynomials in the variables x_1, x_2, x_3, and x_4 given by subscript permutation:

$$\sigma.f(x_1, x_2, x_3, x_4) := f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

for $\sigma \in \mathfrak{S}_4$ and $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$. Calculate the decomposition of V into a direct sum of irreducible $\mathfrak{S}_4$-modules and determine the structure (as a product of matrix rings over $\mathbb{C}$) of the endomorphism algebra $\mathrm{End}_{\mathfrak{S}_4}(V)$.

6. (25 points) Give examples of each of the following objects.

(a) A finite group G, an irreducible G-module V *defined over the real numbers* $\mathbb{R}$, and a G-module homomorphism $\varphi : V \to V$ which is *not* multiplication by a scalar.

(b) An *infinite* group G and a G-module V defined over the complex numbers $\mathbb{C}$ which is indecomposable but not irreducible.

7. (25 points) Consider the following five polynomials $f_1, \ldots, f_5$ in the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4]$ (which are written with respect to the lexicographic term order $<$):

$$f_1 = x_1^2, \;\; f_2 = x_2^2, \;\; f_3 = x_3^2, \;\; f_4 = x_4^2, \;\; f_5 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

Let $I = \langle f_1, \ldots, f_5 \rangle$ be the ideal generated by these polynomials.

(a) It can be shown that $G = \{f_1, \ldots, f_5\}$ is a Gröbner basis for I with respect to $<$. Describe the procedure (Buchberger's Criterion) which verifies this. *You do not need to do this procedure.*

(b) Describe a vector space basis for the quotient $\mathbb{Q}[x_1, x_2, x_3, x_4]/I$ consisting of images $m + I$ of monomials $m \in \mathbb{Q}[x_1, x_2, x_3, x_4]$.

8. (25 points) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in GL_2(\mathbb{C})$ and let G be the cyclic subgroup of $GL_2(\mathbb{C})$ of order 4 generated by A.

(a) Calculate the Hilbert series of the invariant ring $\mathbb{C}[x, y]^G$.

(b) Describe a finite set of polynomials which generates $\mathbb{C}[x, y]^G$ as a $\mathbb{C}$-algebra (you need not compute this set explicitly).