Applied Algebra Qualifying Exam: Part A

5:00pm–8:00pm (PDT), via Zoom. Meeting ID: 912 8480 3260
Tuesday May 11th, 2021

- Write your name and student PID at the top right corner of each page of your submission.
- Do all four problems. Show your work.
- This part of the exam will represent 40% of the total score.
- Your completed examination must be uploaded to Gradescope while you are connected to Zoom. You may leave the meeting once the Proctor has checked that your exam has been uploaded.
- It is your responsibility to check that any uploaded material is both complete and legible.
- By participating in this exam you are agreeing to abide by the UCSD Policy on Academic Integrity. The instructors reserve the right to require a follow-up oral examination.
- This is a closed-book examination. No cell-phone or Internet aids.
- Please keep your camera turned on throughout the exam.
- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex components.
  - $\mathcal{M}_n$ denotes the set $\mathcal{M}_{m,n}$ with $m = n$.
  - $\mathbb{C}^n$ is the set of column vectors with n complex components.
  - x^H is the Hermitian transpose of a vector or matrix x.
  - $\text{eig}(A)$ is the set of eigenvalues of the matrix A (counting multiplicities).

Question 1.

(a) (4 points) State, *but do not prove*, the Schur decomposition theorem for a matrix $A \in M_n$.

(b) (8 points) Let (λ, x) be a simple eigenpair of $A \in M_n$ with $x^H x = 1$. Prove that there exists a nonsingular matrix $\begin{pmatrix} x & X \end{pmatrix}$ with inverse $\begin{pmatrix} y & Y \end{pmatrix}^H$ such that

$$\begin{pmatrix} y^H \\ Y^H \end{pmatrix} A \begin{pmatrix} x & X \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

(c) (8 points) Hence prove that the angle θ between x and y satisfies $\sec\theta = \|y\|_2$.

Question 2.

(a) (8 points.) Consider any Hermitian $A \in \mathcal{M}_n$ with eigenvalues ordered so that $\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A)$. Prove that

$$\lambda_n = \min_{x \neq 0} \frac{x^H A x}{x^H x}.$$

(b) (12 points) Suppose that $D \in \mathcal{M}_n$ with $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. Prove that for all $1 \leq p \leq \infty$ the p-norm of D is given by $\|D\|_p = \max_{1 \leq i \leq n} |d_i|$.

Question 3.

(a) (4 points.) State, *but do not prove*, the singular-value decomposition theorem.

(b) (8 points.) For a given $A \in \mathcal{M}_{m,n}$, prove that

$$\sigma_1(A) = \max_{x,y \neq 0} \frac{|y^H A x|}{\|y\|_2 \|x\|_2},$$

where $\sigma_1(A)$ is the largest singular value of A.

(c) (8 points.) For any $A \in \mathcal{M}_n$, define (i) the field of values $\mathcal{F}(A)$; (ii) the spectral radius $\rho(A)$; and the numerical radius $\omega(A)$. Prove that $\rho(A) \leq \omega(A) \leq \sigma_1(A)$.

Question 4.

(a) (8 points) Let $C \in \mathcal{M}_{m,n}$ with $\text{rank}(C) = m$. Find orthogonal projections that project $x \in \mathbb{C}^n$ onto $\text{range}(C^H)$ and $\text{null}(C)$. Verify that your projections satisfy the properties of an orthogonal projection.

(b) For a given nonzero $y \in \mathbb{C}^n$, let $\mathcal{Y} = \text{span}(y)$.

- **(i)** (6 points) Find an *oblique* projector A that project vectors onto $\mathcal{Y}$. Find the complementary projection.
- **(ii)** (6 points) Find the unique orthogonal projector A that projects vectors onto $\mathcal{Y}$. Find the complementary projection associated with A.

Applied Algebra Qualifying Exam: Part B
Spring 2021

Instructions: Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

Problem 1: Let G be a group (possibly infinite) and let V be a finite-dimensional G-module over $\mathbb{C}$. Assume that V admits a G-invariant inner product $\langle -, - \rangle$. Prove that V is completely reducible.

Problem 2: Let $\mathbb{R}_+$ be the group of positive real numbers under multiplication. Is every indecomposable $\mathbb{R}_+$-module over the complex numbers irreducible?

Problem 3: Let $\lambda, \mu \vdash n$ be partitions and let S^λ, S^μ be the corresponding irreducible S_n-modules. Endow the tensor product $S^\lambda \otimes S^\mu$ with the structure of an S_n-module by the rule

$$\sigma \cdot (v \otimes w) := (\sigma \cdot v) \otimes (\sigma \cdot w)$$

for $\sigma \in S_n, v \in S^\lambda, w \in S^\mu$. Find the vector space dimension of the S_n-fixed subspace

$$(S^\lambda \otimes S^\mu)^{S_n}$$

of $S^\lambda \otimes S^\mu$.

Problem 4: Find the character table of the alternating subgroup A_4 of the symmetric group S_4. The group algebra of A_4 is isomorphic to a direct sum

$$\mathbb{C}[A_4] \cong \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_r}(\mathbb{C})$$

of matrix algebras over $\mathbb{C}$. Determine r and the numbers $n_1, \ldots, n_r > 0$.

Applied Algebra Qualifying Exam: Part C

5:00pm–8:00pm (PDT), via Zoom. Meeting ID: 912 8480 3260
Tuesday May 11th, 2021

- Write your name and student PID at the top right corner of each page of your submission.
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Question 1.

(a) (2 points) Let $\mathrm{B}(n)$ be the permutation group generated by the transpositions $\tau_i = (2i-1\ 2i),\ 1 \le i \le n$. Show that every character χ of $\mathrm{B}(n)$ takes values in $\{-1, 1\}$.

(b) (8 points) Explicitly describe the dual group of $\mathrm{B}(n)$.

Question 2.

(a) (2 points.) With notation as in the previous problem, give the definition of the Cayley graph of $\mathrm{B}(n)$ as generated by $\tau_1, \ldots, \tau_n$.

(b) (8 points) Compute the eigenvalues and eigenvectors of the adjacency operator of the graph in part (a).