Complex Analysis Qualifying Exam May 20th, 2024

Name (PRINT):__________________________________ PID:_______________

Signature: _______________________________________

Instructions: This is a closed-book examination. You have 180 minutes to complete the test. You may use without proof results proved in Conway up to and including Chapter XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail. If using a homework problem, please make sure you reprove it.

Notation and terminology: The unit disk $\{|z| < 1\}$ is denoted by $\mathbb{D}$. A region is an open and connected subset of $\mathbb{C}$.

Problem	Points	Score
#1	10	
#2	10	
#3	10	
#4	10	
#5	10	
#6	10	
#7	10	
Total	70	

1. [10] Let f be a holomorphic function on a region G such that $|f|^2 + |f|$ is harmonic on G. Prove f is constant.

2. [10] Let $f : \mathbb{C} \to \mathbb{C}$ be an entire nowhere zero function. Define $U = \{z : |f(z)| < 1\}$. If $U \neq \emptyset$, show that the connected components of U are unbounded.

3. [10] Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Assume $\text{Re } f(z) > 0$ for all $z \in \mathbb{D}$. Show that

$$|f'(0)| \leq 2 \, \text{Re } f(0).$$

4. [10] Let ϕ be a positive harmonic function on a simply connected region G. Prove that there are two harmonic functions u, v on G such that $\phi = e^u \sin v$.

5. [10] Show that there exist polynomials p_n such that

(i) $p_n(0) = 1$, $p_n'(0) = 0$,

(ii) $p_n(z) \to 0$ as $n \to \infty$ for all fixed $z \in \mathbb{C} \setminus \{0\}$.

6. [4, 6] Let $U \subset \mathbb{C}$ be a bounded connected open set containing 0, and $f : U \to U$ a holomorphic function which satisfies $f(0) = 0$ and $|f'(0)| < 1$. Write

$$f^{(n)} = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}.$$

(i) Show that there is a neighborhood V of 0 such that the sequence $f^{(n)}$ converges to 0 locally uniformly on V.

Hint: $|f(z)| \leq M|z|$ for a constant $M < 1$, for $|z|$ small.

(ii) Show that the sequence $f^{(n)}$ converges locally uniformly to 0 on U.

7. [2, 4, 4] Let $U \subset \mathbb{C}$ be an open set. $f : U \setminus \{a\} \to \mathbb{C}$ be a holomorphic function with an isolated singularity at $a \in U$.

Let P be a non-constant polynomial. Let $g : U \setminus \{a\} \to \mathbb{C}$ be given by

$$g(z) = P(f(z)).$$

Show that:

(i) If f has a removable singularity at a, then g has a removable singularity at a.

(ii) If f has a pole at a, then g has a pole at a.

(iii) If f has an essential singularity at a, then g has an essential singularity at a.