ALGEBRA QUALIFYING EXAM, SPRING 2022

All problems are worth 15 points. You may quote theorems from class or the course notes, unless this trivializes the problem, or if you are asked to reproduce the proof of such a theorem rather than quote it. You may use the result of one part of a problem in the proof of a later part, even if you do not solve the earlier part.

1. Let G be a group of order 2022, which has prime factorization $2022 = (2)(3)(337)$. Suppose that the Sylow 2-subgroup of G is normal. Classify such groups G up to isomorphism.

2. (a) Prove that for $n \geq 5$, the alternating group A_n has no subgroup H such that $|A_n : H| = n/2$. Conclude that A_n has no subgroup isomorphic to S_{n-1}.

(b) Is the result of part (a) true when $n = 4$?

3. Let R be a PID and let M be a finitely generated R-module. Let K be the field of fractions of R.

Suppose that you do not know M but are given the value of $\dim_K(M \otimes_R K)$ as well as $\dim_{R/\mathfrak{m}}(M \otimes_R (R/\mathfrak{m}))$ for each maximal ideal $\mathfrak{m}$ of R. Which of the following can you determine precisely from this information? If the data can be determined, explain how, and if not, give 2 examples of M to show that the calculation is ambiguous.

(a) the rank of M.

(b) the number of elementary divisors of M that are powers of a given prime p.

(c) the exact list of elementary divisors of M.

4. Let F be an algebraically closed field. For which $m \geq 1$ is it true that every matrix $A \in M_2(F)$ such that $A^m = I$ is diagonalizable? (The answer may depend on the characteristic of F.)

5. Let $\zeta \in \mathbb{C}$ be a primitive nth root of unity for $n \geq 3$. Let $K = \mathbb{Q}(\zeta)$.

Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_n^\times$, the group of units of integers mod n. (This is a basic result that was proved during the course; you should reprove it. You may assume that the nth cyclotomic polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}$).

Date: May 18, 2022.

6. Let A be a commutative ring and let $f_1, \dots, f_r \in A$ be elements which generate the unit ideal.

(a) Show that for any non-negative integers $p_1, \dots, p_r$, the powers $f_1^{p_1}, \dots, f_r^{p_r}$ also generate the unit ideal.

(b) Let M be an A-module. Show that $M = 0$ if and only if $M_{f_i} = 0$ for $i = 1, \dots, r$. (Recall that M_f is the localization of M at the multiplicative system $\{1, f, f^2, f^3, \dots\}$.)

7. Let k be a field and let A be a finitely generated commutative k-algebra.

(a) Suppose that A is a simple as an A-module. Prove that A is a finite-dimensional k-vector space.

(b) Show that A is an artinian ring if and only if A is a finite-dimensional k-vector space.