Algebra Qualifying Exam - Spring 2011

Problem 1. (15 pts)

Let G be a group with 117 elements which contains an element of order exactly 9. Classify all such groups G up to isomorphism.

Problem 2. (10 pts)

Let G be a finite group and let p be the smallest prime dividing $|G|$. Assume that G has a unique subgroup H of order p. Show that H is contained in the center of G.

Hint: For each $g \in G$, prove that the permutation $\sigma_g(h) = ghg^{-1}$ of the set $H \setminus \{e\}$ is trivial by investigating its order.

Problem 3. (10 pts)

Consider a commutative ring A with unity and let $\mathfrak{n}$ be its nilradical. Show that the following statements are equivalent:

(i) A has only one prime ideal
(ii) every element in A is either a unit or nilpotent
(iii) $A/\mathfrak{n}$ is a field.

Problem 4. (15 pts)

Let K denote a splitting field of $x^7 - 1$ over $\mathbb{Q}$ inside $\mathbb{C}$. Determine all subfields of K. Express each subfield in the form $\mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{C}$, where you must justify that α is a primitive element for the subfield if it is not obvious.

Problem 5. (15 pts)

Let k be an algebraically closed field of characteristic $p > 0$ and let $K = k(t)$ be a purely transcendental extension in one indeterminate t. Let $n \geq 1$ be any integer, and let L be the splitting field of the polynomial $x^n - t$ over K. It may be helpful in this problem to write $n = p^i m$ where $\gcd(m, p) = 1$.

(i) (7 pts) Show that $L = K(\alpha)$ where α is any root of $x^n - t$ in L.
(ii) (8 pts) Let $G = \mathrm{Aut}(L/K)$ be the group of all automorphisms of L fixing K pointwise, and let $F = \mathrm{Fix}(G)$ be the subfield of L of elements fixed by G. Calculate $[F : K]$.

Problem 6. (15 pts)

Given vector spaces V and W over the complex numbers, suppose that $\phi : V \to V$ and $\psi : W \to W$ are $\mathbb{C}$-linear transformations.

(i) (6 pts) Show that there is a unique linear transformation

$$\phi \otimes \psi : V \otimes_{\mathbb{C}} W \to V \otimes_{\mathbb{C}} W$$

with the property that

$$(\phi \otimes \psi)\,(v \otimes w) = \phi(v) \otimes \psi(w)$$

for all $v \in V$, $w \in W$.

(ii) (9 pts) Let V and W be finite-dimensional of complex dimensions m and n respectively. Prove that

$$\det(\phi \otimes \psi) = \det(\phi)^n \det(\psi)^m.$$

Hint: Choose $\mathbb{C}$*-bases for* V *and* W *such that the matrices representing* ϕ *and* ψ *have a special form.*

Problem 7. (10 pts)

Let R be an integral domain. Prove or give an example to disprove (with justification):

(i) (3 pts) If M is a torsion R-module, then $\mathrm{Ann}_R(M) \neq 0$.
(ii) (3 pts) If M is a free R-module, then M is torsionfree.
(iii) (4 pts) If M is a torsionfree R-module, then M is free.

Problem 8. (15 pts)

Assume A is a commutative ring with unity, and let $f_1, \ldots, f_n \in A$ generate the unit ideal (1). Assume that the rings of fractions $A_{f_1}, \ldots, A_{f_n}$ are Noetherian. Prove that A is Noetherian.