Algebra qualifying exam. 5/22/2009

Instructions

Do as many problems as you can, as completely as you can. You are not necessarily expected to finish the whole exam. However, you should work on at least some problems from each of the three sections (group theory, field theory, ring and module theory.)

If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. If your argument depends on a significant theorem, say what result you are using.

Group theory

1. (15 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x + 1$. Let G be the subgroup generated by f and g of the group of all permutations of the set $\mathbb{R}$. Prove carefully that

$$G \cong \langle a, b \mid a^2 = e, ab = b^{-1}a \rangle.$$

2. (20 points) In this problem, let G be a group of order $105 = 3 \cdot 5 \cdot 7$.

(a). Suppose that G does not have a normal Sylow 7-subgroup. Show in this case that G has a normal Sylow 3-subgroup and a normal Sylow 5-subgroup. Prove then that G is abelian, a contradiction.

(b). Show that there is a non-abelian group G of order 105. Explain why your group G is solvable, but not nilpotent.

Field theory

3. (15 points) Let p be a prime number and let K, L be fields of orders p^m, p^n respectively, where $m < n$. When is K isomorphic to a subfield of L?

4. (20 points) Let $a = \sqrt{2 + \sqrt{2}}$ in $\mathbb{C}$ and let f be the minimal polynomial of a over $\mathbb{Q}$. Let E be the splitting field for f over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.

5. (15 points) Let E/F be a Galois extension and let K, L be intermediate fields. Show that K and L are F-isomorphic (i.e. there exists an isomorphism from K to L, which is the identity on F) if and only if the subgroups of $G = \mathrm{Gal}(E/F)$ corresponding to K and L are conjugate in G.

Ring and module theory

6. (15 points) A commutative ring R with identity is *local* if it has exactly one maximal ideal. In this problem, let R be a local ring which is a PID, but is not a field.

(a). Classify the possible finitely generated R-modules, in terms of their invariant factors (or elementary divisors). Show that there is only a countably infinite number of finitely generated R-modules, up to isomorphism.

(b). Prove that if M and N are nonzero, finitely generated, torsion R-modules, then there exists a nonzero R-module homomorphism $\phi : M \to N$.

7. (20 points) (a). Show that a commutative Noetherian domain R has the following property: given any nonzero, nonunit $x \in R$, one has $x = z_1 z_2 \ldots z_m$ for some irreducible elements $z_i \in R$.

(b). Prove that any UFD is integrally closed in its field of fractions.

(c). Give, with proof, an example of a domain R which is Noetherian, but not a UFD.

8. (15 points) (a). Consider the ideal $I = (x^2 - y^3, x - y^2) \subseteq \mathbb{C}[x, y]$. Find $\operatorname{rad} I$, the radical of I, expressing it as an intersection of prime ideals (do not try to find a generating set for $\operatorname{rad} I$.)

(b). Is I a radical ideal?

Applied Algebra Qualifying Exam

May 29. 2009

Part 1: Matrix Theory

#1	25	
#2	20	
#3	15	
Total	60	

1. (25) (a) State and prove the Schur Decomposition Theorem.

   (b) Use Schur to prove that a square matrix A has an orthonormal basis of eigenvectors iff $A^H A = AA^H$.

2. (20) (a) Using 2×2 matrices. show that the Jordan Canonical Form of a matrix is NOT a continuous function of the elements of the matrix.

   (b) Show that every real quadratic form over the complexes, (i.e., for every $z \in \mathbb{C}^n$, $f(z) = z^H Az \in \mathbb{R}$) can be generated by a Hermitian matrix.

   (c) Show that if $z^H Az = 0$ for all $z \in \mathbb{C}^n$, then $A = 0$.

   (d) Show by 2×2 example that $x^T Ax = 0$ for all $x \in \mathbb{R}^n$, where A is real, does not imply that $A = 0$.

3. (15) Let $\hat{x}$ be a least squares solution to $Ax = b$, where A is $m \times n$ and $m \geq n$. Let $A^\dagger$ be the pseudo-inverse of A. Use the Singular Value Decomposition to show that $\tilde{x} = A^\dagger b$ is the min 2-norm least squares solution to $Ax = b$, i.e., show

   (a) $\tilde{x}$ is a least squares solution,

   (b) if $\hat{x}$ is a least square solution then $||\hat{x}||_2 \geq ||\tilde{x}||_2$, and

   (c) $\tilde{x}$ is unique.

APPLIED ALGEBRA QUALIFYING EXAM – SPRING 2009

This part of the Applied Algebra exam will be scaled to make up 60% of the whole exam. The problems have the same value, except for the last one which will count more. Try to do as many problems as possible.

1. Let $p_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ and let d be the diagonal matrix with diagonal entries $x_1, \ldots x_N$, where $V = \mathbf{C}^N$. The matrix d acts on each factor of $V^{\otimes n}$, thereby defining a linear action on $V^{\otimes n}$. The action of S_n on $V^{\otimes n}$ is given via permuting the tensor factors.
(a) Calculate $Tr_{V^{\otimes n}}(p_n d)$.
(b) The value of $Tr_{V^{\otimes 10}}((p_4 \otimes p_4 \otimes p_2)d)$ can be written as a linear combination of Schur functions. Calculate the coefficient of $s_{[6,3,1]}$.
(c) Calculate the multiplicity of the simple S_{10} module $S^{[3,3,3,1]}$ in $V^{\otimes 10}$ for $\dim(V) = N = 5$ and for $N = 3$.

2. Let e_r be the r-th elementary symmetric function in the variables $x_1, x_2, \ldots, x_n$.
(a) Show that the determinant of $(\partial e_i/\partial x_j)_{1\leq i,j\leq n}$ is a homogeneous polynomial and calculate its degree.
(b) Calculate the determinant.

3. Let $\rho : G \to Gl(V)$ be a representation of the finite group G into the group $Gl(V)$ of invertible linear maps on the vector space V.
(a) Show that also the map $\hat\rho : g \in G \mapsto \rho(g^{-1})^t$ defines a representation, where ^{t} means the transpose of a matrix.
(b) Let χ_ρ and $\chi_{\hat\rho}$ be the characters of ρ and $\hat\rho$. Show that $\chi_{\hat\rho}(g) = \bar\chi_\rho(g)$ (i.e. the complex conjugate) for all $g \in G$.
(c) Let V be a simple G-module. Show: If W is a simple G module such that the trivial representation occurs in $V \otimes W$, then W must be isomorphic to the representation defined in (a).

4. Let $f_1 = x^2y^2 - x$ and $f_2 = x^3y - 1$.
(a) Calculate a Gröbner basis for $\langle f_1, f_2\rangle \cap k[x]$ and for $\langle f_1, f_2\rangle$, where $\langle f_1, f_2\rangle$ is the ideal in $k[x, y]$ generated by f_1 and f_2.
(b) What is the variety $V(f_1, f_2) = \{(a, b),\ f(a, b) = 0, f \in \langle f_1, f_2\rangle\}$ for $k = \mathbf{C}$?

5. Let G be the subgroup of S_4 generated by the permutations (12) and (34), and let V be the simple representation of S_4 labeled by the Young diagram [2, 1, 1].
(a) Calculate the Molien series of $k[x_1, x_2, x_3]^G$.
(b) Let $\tilde G \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ with generators g_1 and g_2, and let W be the three-dimensional $\tilde G$ module with basis w_1, w_2, w_3 such that the action of G is given by

$$g_1 w_i = \begin{cases} -w_i & \text{if i=1,2,} \\ w_3 & \text{if i=3,} \end{cases} \qquad g_2 w_i = \begin{cases} w_1 & \text{if i=1,} \\ -w_i & \text{if i=2,3.} \end{cases}$$

Write down a Hironaka decomposition for $k[y_1, y_2, y_3]^{\tilde G}$.
(c) Find presentations of the rings $k[y_1, y_2, y_3]^{\tilde G}$ and $k[x_1, x_2, x_3]^G$ via generators and relations.