Algebra qualifying exam May 25, 2007

Name: ________________________________

"[n]" means the problem is worth n points.

1. Let p be a prime, and G a group of order p^3.
a [20]. Prove that G has a normal subgroup of order p^2.

b. Assume that G has a *cyclic* normal subgroup N of order p^2, generated by some element n. Let g be an element not in N.

i [5]. If the order $|g|$ of g is p^3, classify the possible G up to isomorphism.

ii [15]. If the order $|g|$ of g is p, classify the possible G up to isomorphism.

(Incidentally, there exist groups of neither type, such as the group of 3×3 upper triangular matrices over $\mathbb{F}_p$ with 1s on the diagonal.)

2. Let I, J be two ideals in a commutative ring R (with unit).
a [20]. Define $K = \{r : rJ \leq I\}$. Show that K is an ideal.

b [10]. If R is a principal ideal domain, so $I = \langle i \rangle$, $J = \langle j \rangle$, give a formula for a generator k of K.

3 [25]. Describe, up to isomorphism, all the $\mathbb{R}[x]$-module structures one might put on a 3-dimensional real vector space (extending the fixed $\mathbb{R}$-action).

4. Let $\mathbb{C}[x]/(x^n)$ denote the evident $\mathbb{C}[x]$ (bi)module, and let $m, n \in \mathbb{N}$.
a [15]. Show that there exist $d_1, \ldots, d_k$ such that

$$\mathbb{C}[x]/(x^n) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x^m) \cong \bigoplus_{i=1}^{k} \mathbb{C}[x]/(x^{d_i}).$$

b [20]. Determine the $\{d_i\}$ in terms of m, n.
Hint: figure out the action of x on the obvious $\mathbb{C}$-basis.

5 [30]. Recall that a "perfect" field of characteristic p is one for which the Frobenius map $Fr : x \to x^p$ is onto.
Let K be a perfect field, and F an algebraic extension. Show that F is perfect.