May 2004

QUALIFYING EXAM
ALGEBRA
Parts II and III

Part II

1.(10 points) Prove the 3rd Sylow Theorem. Suppose p is a prime dividing the order of a group G. Then the number of p-Sylow subgroups divides the order of G and is congruent to $1 mod p$. You may use the first two Sylow Theorems without proof.

2.(20 points) Let G be the group of 2×2 invertible matrices with entries in the finite field $\mathbb{Z}_p$. Then we know that $|G| = (p-1)^2 p(p+1)$. Assume that $p = 17$. Then $|G| = 2^9 3^2 17$.

a. Let x be an element of G of order 17. Prove that x is conjugate to an element of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

b. Prove that G contains 18 Sylow 17-subgroups. Hint: Use the fact that the upper triangular matrices contain a Sylow 17-subgroup as a normal subgroup.

c. How many elements in G have order 17?

3.(10 points) Construct a nonabelian group of order $75 = 5^2 \cdot 3$.

Part III

4.(20 points) Let n_i, $0 \leq i \leq m$ be integers. Use the Chinese Remainder Theorem to prove there exists a unique polynomial $f(X) \in \mathbb{Q}[X]$ of degree $\leq m$ with $f(i) = n_i, 0 \leq i \leq m$.

5.(10 points) In a commutative ring with 1 prove that every ideal is contained in a maximal (proper) ideal.

6.(20 points) Prove that a projective R-module is flat. Hint: First prove the case for a free R module.

7.(20 points)

a. Determine the Galois group G for the splitting field K of $X^5 - 3$ over $\mathbb{Q}$.

b. Determine all the subgroups of G isomorphic to $\mathbb{Z}_5$

8.(20 points) Let $\Phi = X^2 + X + 1$ be the third cyclotomic polynomial.

a. Prove that Φ is reducible over $\mathbb{Z}_7$ and give a complete factorization.

b. Determine the possible rational canonical forms for any element $A \in GL(2, \mathbb{Z}_7)$ which satisfies $A^3 = 1$.

9.(10 points) Let$V$and W be finite dimensional vector spaces over the field of complex numbers of dimensions m and n. Use the universal mapping property to prove that $V \otimes W$ is a vector space of dimension mn.

10.(10 points) In the Gaussian integers the norm is used to analyze sums of squares. Use this technique to determine how many ways $N = 3^4$ and $M = 5^4$ can each be expressed as the sum of two integer squares.