SPRING 2014 UCSD ALGEBRA QUALIFYING EXAM

Instructions: Do as many problems as you can, as completely as you can. If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. You may quote theorems proved in class or in the textbook, unless the point of the problem is to reproduce the proof of that theorem.

(1) (10 pts) Let G be a finite p-group. Show that, if H is a proper subgroup of G, then H is a proper subgroup of its normalizer $N_G(H)$. *(Hint: center of a finite p-group is non-trivial.)*

(2) Let G be a finite group. Suppose for any p-subgroup Q of G, there is a unique Sylow p-subgroup P which contains Q. Moreover, assume G has a normal subgroup N such that $p||N|$.
  - (a) (5 pts) Show that if P_1 and P_2 are two Sylow p-subgroups, then $nP_1n^{-1} = P_2$ for some $n \in N$. *(Hint: Consider a Sylow p-subgroup of N.)*
  - (b) (5 pts) Show that G/N has a unique Sylow p-subgroup.

(3)
  - (a) (7 pts) Show that $\mathbb{Z}[i]$ is a PID.
  - (b) (3 pts) Let A be a proper subring of $\mathbb{Z}[i]$ which contains $\mathbb{Z}$. Suppose the field of fractions of A is $\mathbb{Q}(i)$. Show that A is NOT a unique factorization domain (UFD).

(4) (10 pts) Let A be a unital commutative ring which is Artinian, i.e. if $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ is a chain of ideals of A, then $\mathfrak{a}_{k_0} = \mathfrak{a}_{k_0+1} = \cdots$ for some k_0.
  - (a) Show that, if A is an integral domain, then A is a field. *(Hint: Consider $\langle x \rangle \supseteq \langle x^2 \rangle \supseteq \cdots$.)*
  - (b) Prove that $\mathfrak{p} \lhd A$ is a prime ideal if and only if $\mathfrak{p}$ is a maximal ideal.

(5) (10 pts) Prove that an abelian group A is injective as a $\mathbb{Z}$-module if and only if it is divisible.

(6)
  - (a) (7 pts) Let $F \subseteq E$ be a field extension, and $f(x) \in F[x]$. Prove that
$$E \otimes_F F[x]/\langle f(x) \rangle \simeq E[x]/\langle f(x) \rangle$$
as E-algebras. (Notice that $\langle f(x) \rangle$ at the left hand side is $F[x]f(x)$ and in the right hand side is $E[x]f(x)$). *(Hint: $\{\overline{x}^i | 0 \leq i \leq \deg f - 1\}$ is an F-basis of $F[x]/\langle f(x) \rangle$.)*
  - (b) (8 pts) Let E/F be a finite Galois extension. Prove that
$$E \otimes_F E \simeq \underbrace{E \oplus \cdots \oplus E}_{[E:F] \text{ times}}$$
as E-algebras.

(7) (10 pts) Let L/K be a Galois extension with Galois group S_3. Prove or disprove that L is the splitting field of an irreducible cubic polynomial with coefficients in K.

(8) (10 pts) In the factorization of $X^{64} - X$ into irreducible terms, how many terms of each degree appear over the field $K = \mathbb{F}_2$? (e.g. 5 terms of degree 1, 4 terms of degree 7, etc...) Justify your answer, but you do not have to actually find the factorization.

Good luck!