Name: ______________________________

PID: ______________________________

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	15	
6	15	
7	10	
Total:	85	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even if you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is to reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Suppose G is a finite group and P is a p-subgroup of G which is not a Sylow p-subgroup. Prove that p divides $|N_G(P)/P|$.

2. For any finite group H and prime ℓ, we denote the set of Sylow ℓ-subgroups of H by $\mathrm{Syl}_\ell(H)$. Suppose $p < q$ are odd primes, G is a finite group, and $|\mathrm{Syl}_p(G)| = p + 1$ and $|\mathrm{Syl}_q(G)| = q + 1$.

   (a) (5 points) Suppose $P \in \mathrm{Syl}_p(G)$. Prove that G has a Sylow q-subgroup Q such that $Q \subseteq N_G(P)$.

   (b) (5 points) In the setting of part (a), let $H = PQ$. Argue why H is a subgroup and prove that either $Q \trianglelefteq H$ or $|\mathrm{Syl}_q(H)| = q + 1$.

   (c) (5 points) In the setting of part (b), prove that $H \simeq P \times Q$.

3. (10 points) Suppose A is a unital commutative ring. Let $A[x]$ be the ring of polynomials, $A^{\times}$ be the group of units of A, and $\mathrm{Nil}(A)$ be the nilradical of A. Prove that

$$A[x]^{\times} = \{a_0 + a_1 x + \cdots + a_n x^n |\, a_0 \in A^{\times}, a_1, \ldots, a_n \in \mathrm{Nil}(A), n \in \mathbb{Z}^{+}\},$$

where $A[x]^{\times}$ is the group of units of $A[x]$. (**Hint.** Without proof you can use that $\sum_i a_i x^i \mapsto \sum_i (a_i + \mathfrak{a}) x^i$ is a ring homomorphism from $A[x]$ to $(A/\mathfrak{a})[x]$ for any $\mathfrak{a} \trianglelefteq A$. Show that if u is a unit and n is nilpotent, then $u + n = u(1 + u^{-1} n)$ is a unit.)

4. Suppose F is a field, $a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in M_3(F)$, and $F[a]$ is the subring generated by F and a. Suppose $\overline{F}$ is an algebraic closure of F.

   (a) (6 points) Prove that $F[a] \otimes_F \overline{F} \simeq \overline{F} \oplus \overline{F} \oplus \overline{F}$ as $\overline{F}$-algebras if the characteristic of F is not 3.

   (b) (4 points) Prove that $F[a] \otimes_F \overline{F} \simeq \overline{F}[x]/\langle x^3 \rangle$ as $\overline{F}$-algebras if the characteristic of F is 3.

5. Suppose $D = \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{a} = \langle 3, 1+\sqrt{-5}\rangle$.

(a) (5 points) Prove that $\mathfrak{a}$ is not a principal ideal.

(b) (10 points) Prove that $\mathfrak{a}$ is a projective D-module.

6. Suppose ℓ and p are prime and $\mathbb{F}_p$ is a finite field of order p.

   (a) (5 points) Prove that the degree of an irreducible factor of $x^{p^\ell} - x$ is either 1 or ℓ.

   (b) (5 points) Suppose $f(x) \in \mathbb{F}_p[x]$ is a monic irreducible polynomial of degree ℓ. Prove that $f(x)|(x^{p\ell} - x)$.

(c) (5 points) Prove that there are exactly $\frac{p^\ell - p}{\ell}$ many monic irreducible polynomials of degree ℓ in $\mathbb{F}_p[x]$.

7. For a positive integer n, let $\zeta_n = e^{2\pi i/n}$ be a primitive n-th root of unity in $\mathbb{C}$.

   (a) (2 points) Prove that $\mathbb{Q}[\sqrt[5]{5}, \zeta_5]$ is a splitting field of $x^5 - 5$ over $\mathbb{Q}$.

(b) (4 points) Prove that $\mathbb{Q}[\sqrt[5]{5}, \zeta_5]/\mathbb{Q}$ is a non-abelian Galois extension; that means it is a Galois extension and $\mathrm{Gal}(\mathbb{Q}[\sqrt[5]{5}, \zeta_5]/\mathbb{Q})$ is not abelian.

(c) (4 points) Prove that $x^5 - 5$ does not have a zero in $\mathbb{Q}[\zeta_{25}]$.

Good Luck!