Name: ______________________________

PID: ______________________________

Question	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	15	
7	25	
Total:	100	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even if you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of such a result. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Suppose $p < q$ are two odd primes. Suppose G is a group of order $2pq$. Prove that G has normal subgroups N_1 and N_2 such that $|N_1| = pq$, $|N_2| = q$, and $N_2 \subseteq N_1$.

2. Suppose p is a prime which is at most n, and F is a field of characteristic p. Suppose $g \in \mathrm{GL}_n(F)$ and $g^{p^m} = I$ for some positive integer m.

   (a) (5 points) Prove that $g - I$ is a nilpotent matrix.

   (b) (5 points) Prove that $g^p = I$.

3. Suppose A is a commutative unital ring, and M is an A-module.

   (a) (8 points) Prove that, if $M_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m}$ of A, then $M = 0$.

(b) (7 points) Prove that if $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of A, then M is a flat module. (You do not need to prove that localization is an exact functor.)

4. (10 points) Let A be a unital commutative ring. Suppose P and Q are two projective A-modules. Prove that $P \otimes_A Q$ is a projective A-module.

5. Let $A := \{a_0 + a_2T^2 + a_3T^3 + \cdots + a_nT^n \mid n = 0, 2, 3, \ldots; a_0, a_2, \ldots, a_n \in \mathbb{Z}\}$ (no degree one term) be a subring of the ring $\mathbb{Z}[T]$ of polynomials.

(a) (2 points) Find the field of fractions of A.

(b) (3 points) Show that T is integral over A; that means it is a zero of a monic polynomial in $A[x]$.

(c) (5 points) Is A a UFD?

(d) (5 points) Is there $f(T)$ such that $A = \mathbb{Z}[f(T)]$?

6. Suppose p is prime and $q = p^n$ for some positive integer n. Let $\mathbb{F}_q$ be a finite field of order q and $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. Suppose $\alpha \in \overline{\mathbb{F}}_q$ is a zero of $x^q - x + 1$.

   (a) (5 points) Prove that $\alpha^{q^i} = \alpha - i$ for any positive integer i.

   (b) (5 points) Prove that $|\mathrm{Gal}(\mathbb{F}_q[\alpha]/\mathbb{F}_q)| = p$.

(c) (5 points) Prove that any irreducible factor of $x^q - x + 1 \in \mathbb{F}_q[x]$ has degree p.

7. Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial of degree p where p is prime. Let E be the splitting field of $f(x)$ over $\mathbb{Q}$. Let $\alpha \in E$ be a zero of f, $G := \mathrm{Gal}(E/\mathbb{Q})$, and $H := \mathrm{Gal}(E/\mathbb{Q}[\alpha])$. Suppose H is not trivial.

   (a) (10 points) Prove that $[G : H] = p$ and $\gcd(|H|, p) = 1$.

(b) (5 points) Prove that H is not a normal subgroup of G.

(c) (10 points) Let P be a Sylow p-subgroup of G. Prove that $N_G(P) \neq P$. (Hint: assuming $N_G(P) = P$, deduce that $H = \{g \in G | \, o(g) \neq p\}$ where $o(g)$ is the order of g.)

Good Luck!