Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	7	
6	15	
7	10	
8	13	
Total:	85	

FALL 2014 UCSD ALGEBRA QUALIFYING EXAM

Instructions: Do as many problems as you can, as completely as you can. If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. You may quote theorems proved in class or in the textbook, unless the point of the problem is to reproduce the proof of that theorem.

(1) (10 points) Let G be a (not necessarily finite) nilpotent group. Prove that for any proper subgroup H we have that $H \neq N_G(H)$.

(2) (10 points) Suppose G is a finite group with exactly n Sylow-p-subgroups and n is at least 2. Prove that the symmetric group S_n has a subgroup with exactly n Sylow-p-subgroups.

(3) (10 points) Let A be a Noetherian unital commutative ring. Prove that there is a finite collection of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that

$$\mathfrak{p}_1 \cdots \mathfrak{p}_n = (0).$$

(**Hint**: *consider the set of all ideals of A which do not contain a finite product of prime ideals.*)

(4) Let $A = \mathbb{Z}[\sqrt{10}]$.
  (i) (5 points) Prove that 2 is irreducible in A.
  (ii) (3 points) Prove that 2 is NOT prime in A.
  (iii) (2 points) Is A a UFD? Justify your answer.

(5) (i) (4 points) Suppose $x \in M_n(\mathbb{C})$ is a nilpotent matrix. Prove that $x^n = 0$. (**Hint**: *think of the characteristic polynomial.*)
  (ii) (3 points) How many similarity classes are there of nilpotent matrices in $M_4(\mathbb{C})$?

(6) Let A be a PID.
  (i) (8 points) Prove that a finitely generated projective A-module is free.
  (ii) (7 points) Prove that a finitely generated flat A-module is free.

(7) (10 points) Let L/K be a Galois extension with finite Galois group G. Consider the natural action of the group G on L. Let H be a subgroup of G. Prove that there exists an element $\alpha \in L$ whose stabilizer is H.

(8) Let E/F be a field extension. Let $f(x) \in F[x]$ be a polynomial of degree n with n distinct roots $\alpha_1, \ldots, \alpha_n$ in E. Suppose $[F[\alpha_1, \alpha_2] : F] = n(n-1)$.
  (i) (3 points) Prove that $f(x)$ is irreducible over F and $f(x)/(x - \alpha_1)$ is irreducible over $F[\alpha_1]$.
  (ii) (10 points) Suppose $g(x)$ is the minimal polynomial of $\alpha_1 + \alpha_2$ over F. Prove that $g(\alpha_i + \alpha_j) = 0$ for any $1 \leq i < j \leq n$. (**Hint**: *first show that $g(\alpha_1 + \alpha_j) = 0$ for any $2 \leq j \leq n$; remember you can use the first part!*)

Good Luck!