FALL 2013 UCSD ALGEBRA QUALIFYING EXAM

Instructions: Do as many problems as you can, as completely as you can.
If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. You may quote theorems proved in class or in the textbook, unless the point of the problem is to reproduce the proof of that theorem.

(1) (15 pts) Let G be a finite group. Let p be a prime factor of the order $|G|$ of G. Let $\Omega = \{g \in G | g^p = 1\}$ and let P be a Sylow p-subgroup of G.
  (a) (7 pts) Prove that $C_G(P) \cap \Omega$ is a nontrivial p-subgroup of P.
  (Hint: Consider $P\langle g\rangle$ for $g \in C_G(P) \cap \Omega$.)
  (b) (8 pts) Prove that p divides $|\Omega|$.
  (Hint: Use the fact that P acts on Ω by conjugation.)

(2) (10 pts) Let D be an integral domain. Prove that if $D[x]$ is a PID, then D is a field.

(3) (10 pts) Let A be a commutative noetherian ring.
  (a) (7 pts)
  Show that
  every ideal I of A contains a finite product $P_1P_2\dots P_k$ for some k, where the P_i are (not necessarily distinct) prime ideals.
  (Hint: Consider the set of ideals which do not satisfy this property.)
  (b) (3 pts)
  Show that A has at only finitely many minimal prime ideals.

(4) (10 pts) Let A be a commutative ring. Let P and Q be A-modules.
  (a) (5 pts) Prove that $P \oplus Q$ is an injective A-module if and only if both P and Q are injective A-modules.
  (b) (5 pts) Let $A \subseteq B$, where B is a another commutative ring and A is a unital subring of B. Prove that if P is a projective A-module, then $P \otimes_A B$ is a projective B-module.

(5) (10 pts) Let $A \in M_n(F)$, where F is any field. Let $V = F^n$ and let $\phi : V \to V$ be the linear transformation given by the matrix A.
  Show that the following are equivalent:
  (i) charpoly$(A) =$ minpoly(A).
  (ii) The rational canonical form of A is a companion matrix of a polynominal $f \in F[x]$.
  (iii) There exists $v \in V$ such that the elements $\{v, \phi(v), \phi^2(v), \dots \phi^{n-1}(v)\}$ span V over F.

(6) (10 pts) Let p be an odd prime. Let E be a splitting field of $x^p - x + 1$ over $\mathbb{F}_p$.

(a) (3 pts) Prove that, if $\alpha \in E$ is a root of $x^p - x + 1$, then for any $0 \leq i \leq p$ we have $\alpha^{p^i} = \alpha - i$.

(b) (7 pts) Explain why if K is any finite field, then $|K| = \min\{n \mid x^n = x \text{ for all } x \in K\}$. Then prove that $x^p - x + 1$ is irreducible over $\mathbb{F}_p$ and find $|E|$.

(7) (15 pts) let L/K be an algebraic extension. Let

$$\mathrm{K}^{ab} = \{\alpha \in L \,\big|\, K[\alpha]/K \text{ is Galois and } \mathrm{Gal}(K[\alpha]/K) \text{ is abelian}\}.$$

(a) (7 pts) Prove that if $K \subseteq E \subseteq L$, E/K is Galois and $\mathrm{Gal}(E/K)$ is Abelian, then $E \subseteq K^{ab}$.

(b) (8 pts) Prove that K^{ab} is a subfield of L.