Algebra Qualifying Exam Fall 2011

Problem 1. (10 pts)

Consider a simple group G with 60 elements. Show that if G has a subgroup H of order 12 then $G \cong A_5$.

(Any simple group of order 60 *is isomorphic to* A_5*, but obviously you cannot use this fact, unless you prove it.)*

Problem 2. (10 pts)

Let $n \geq 3$ be an integer. Calculate the number of ordered pairs of permutations (σ, τ) in the symmetric group S_n such that $\sigma\tau = \tau\sigma$. Your answer should be a simple formula involving known functions of n.

Problem 3. (10 pts)

Let A be a commutative ring with unity, and assume that the elements $f_1, \ldots, f_n \in A$ generate the unit ideal (1). Show that there exists an injective ring homomorphism

$$\phi : A \to \prod_{i=1}^{n} A_{f_i}.$$

As usual, A_f denotes the localization of A at the set of powers of f.

Problem 4. (10 pts)

Assume that A is a commutative Noetherian ring, and let $a_n \in A$ for $n \geq 0$. Prove that the power series

$$f = \sum_{n=0}^{\infty} a_n x^n$$

is nilpotent if and only if each $a_n \in A$ is nilpotent.

Problem 5. (10 pts)

Let F be a field and $f(x) = x^4 + bx^2 + c \in F[x]$, for some $b, c \in F$.

If K is the splitting field of $f(x)$ over F, prove that the Galois group $\mathrm{Gal}(K/F)$ is isomorphic to a subgroup of the dihedral group D_4 of order 8.

Problem 6. (15 pts)

(a). (10 pts) Suppose that $K \subseteq L \subseteq M$ are fields such that L/K and M/L are Galois, and that every automorphism of L/K extends to an automorphism of M. Prove that M/K is Galois.

(b). (5 pts) Give an example of fields $K \subseteq L \subseteq M$ such that L/K and M/L are Galois, but M/K is not Galois.

Problem 7. (15 pts)

(a). (10 pts) Let M be a (left) module over a commutative ring R and let I be an ideal of R. Prove that

$$(R/I) \otimes_R M \cong M/IM$$

(as R-modules).

(b). (5 pts) Now let R be a PID and let I be a maximal ideal of R. Suppose that M is a finitely generated R-module. Calculate the dimension of $(R/I) \otimes_R M$ as a vector space over the field $K = R/I$ in terms of the elementary divisors (or invariant factors) of M.

Problem 8. (10 pts)

Find the Jordan canonical form for the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(a). (5 pts) Over the complex numbers.

(b). (5 pts) Over the algebraic closure of the field of three elements.