Algebra qualifying exam, 9/10/2009

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Instructions

Do as many problems as you can, as completely as you can. You are not expected to finish the whole exam, though you should work on some problems from each section. If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. If your argument depends on a significant theorem (for example, Hilbert's Nullstellensatz), say so; but you may simply assume very basic techniques and arguments (for example, Lagrange's theorem in group theory). If you are unsure how much detail is needed, please ask.

Note: For $m \geq 1$, $\mathbb{Z}_m$ means the group $\mathbb{Z}/m\mathbb{Z}$ of integers mod m.

Group theory

1. Let $\phi : \mathbb{Z}_m \to \mathrm{Aut}(\mathbb{Z}_n)$ be a homomorphism, for some integers $m, n \geq 2$. Let G be the semidirect product $\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_m$. Find a presentation for G by generators and relations and prove carefully that your presented group is isomorphic to G.

2. Let G be a group with $|G| = p(p+1)$, where p is an odd prime and $(p+1) = 2^n$ is a power of 2. Suppose further that G does *not* have a normal Sylow p-subgroup.

(2a). Show that G has a normal subgroup H of order $p+1$. Show that G is isomorphic to a semidirect product $H \rtimes \mathbb{Z}_p$.

(2b). Prove that the subgroup H in part (a) is Abelian. (Hint: given a nonidentity element $x \in H$, consider the possible size of the centralizer subgroup $C_G(x)$.)

Field theory

3. Let F be a finite field of order q and let E/F be a field extension. Suppose that an element $a \in E$ is algebraic over F. Prove that $[F[a] : F]$ is the smallest positive integer n such that $a^{q^n} = a$ and that it divides every other such positive integer.

4. Let G be any finite group and F any field. Show that there exist fields L and E with $F \subseteq L \subseteq E$, such that E is Galois over L with the Galois group of E/L being isomorphic to G.

5. Consider the splitting field E of the polynomial $f(x) = x^4 - 5$ over $\mathbb{Q}$.

(5a). Find the degree $[E : \mathbb{Q}]$.

(5b). Determine the Galois group of E over $\mathbb{Q}$ as a subgroup of the symmetric group S_4.

Ring and Module theory

6. Let $n \geq 1$ and consider the ring $M_n(\mathbb{C})$ of $n \times n$ matrices with coefficients in $\mathbb{C}$. Suppose that $A \in M_n(\mathbb{C})$ satisfies $A^3 = A$. Let $V = \mathbb{C}^n$, an n-dimensional vector space over $\mathbb{C}$. Thinking of the elements of V as column vectors, consider the linear transformation $\phi : V \to V$ defined by left multiplication by the matrix A. Prove that V decomposes into a direct sum of three $\mathbb{C}$-linear subspaces, say $V = U_1 \oplus U_2 \oplus U_3$, such that given $v \in V$ with $v = u_1 + u_2 + u_3$ where $u_i \in U_i$, then $\phi(v) = u_1 - u_2$.

7. Let I be an ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ for some $n \geq 1$. Prove that the following conditions on I are all equivalent:

i. $\mathbb{C}[x_1, \ldots, x_n]/I$ is a finite-dimensional $\mathbb{C}$-vector space;

ii. $I \cap \mathbb{C}[x_i] \neq 0$ for all $1 \leq i \leq n$;

iii. The set of common zeroes in affine n-space $\mathbb{C}^n$ of all of the polynomials in I is a finite (or empty) set.

8. For simplicity, let R be a commutative ring with identity in this problem. Suppose that I and J are ideals of R such that R/I and R/J are noetherian rings. Prove that $R/(I \cap J)$ is also a noetherian ring.