Department of Mathematics
MA/PhD Qualifying Examination
in Algebra

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1:00–4:00pm. AP&M 7421
Tuesday September 5, 2006

NAME ____________________

#1.1	20	
#1.2	20	
#2.1	20	
#2.2	20	
#3.1	30	
#3.2	20	
#3.3	30	
#4.1	40	
Total	200	

- Do all problems.
- Add your name in the space provided and staple this page to your solutions for Section #1.
- Start your solutions for the questions in Sections #2 and #4 on a fresh page.
- Write your name clearly on every sheet submitted.

1. Linear Algebra

Question 1.1. Assume that (λ, x) is an eigenpair of $A \in M_n$ such that $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda) = 1$. Prove that there exists a nonsingular matrix $(x \quad X)$ with inverse $(y \quad Y)^*$ such that

$$\begin{pmatrix} y^* \\ Y^* \end{pmatrix} A \, (x \quad X) = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

Question 1.2. Given $A \in M_{m,n}$ with $m \geq n$, prove that there exists a unique $U \in M_{m,n}$ with orthonormal columns, and a unique Hermitian positive semidefinite $H \in M_n$ such that $A = UH$.

(*State in detail any auxiliary results that you use.*)

2. Group Theory

Question 2.1. Let G be a group and let $\mathcal{Z}(G)$ denote its center.

(a) Show that if $G/\mathcal{Z}(G)$ is cyclic then $G = \mathcal{Z}(G)$.

(b) Show that if $\mathrm{card}(G) = p^3$, for some prime number p and G is non-commutative then $\mathrm{card}(\mathcal{Z}(G)) = p$.

(c) Construct a non-commutative group G of cardinality (order) 16 whose center $\mathcal{Z}(G)$ is not cyclic.

Note. As usual, $\mathrm{card}(X)$ denotes the cardinality of the set X.

Question 2.2. Let n be an integer. Let G_n be the group given by generators and relations as follows.

$$G_n = \langle x, y \mid x^3 = 1, \; xyx^{-1} = y^n \rangle$$

(a) Provide (with proof) necessary and sufficient conditions on the integer n for the group G_n to be **finite.**

(b) Assuming that G_n is finite, compute its order as a function of n.

3. Ring Theory and Module Theory

Question 3.1. Let X and Y be two independent variables.

(a) Prove that the ring $R_1 := \mathbb{Q}(X)[Y]/((Y^2 + X)^3)$ is a local ring.

(b) Prove that the ring $R_2 := \mathbb{Q}[X, Y]/((Y^2 + X)^3)$ is not a local ring by giving examples (with proof) of two distinct maximal ideals in R_2.

(c) Give an example of a prime ideal in R_2 which is not maximal. Are there such ideals in R_1? Justify your answers.

Note. As usual, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}(X)$ denotes the field of rational functions of variable X with coefficients in $\mathbb{Q}$, $\mathbb{Q}(X)[Y]$ denotes the ring of polynomials of variable Y with coefficients in $\mathbb{Q}(X)$, and $\mathbb{Q}[X, Y]$ denotes the ring of polynomials of variables X, Y with coefficients in $\mathbb{Q}$. Also, recall that a commutative ring is called **local** if it has a unique maximal ideal.

Question 3.2. Consider the $\mathbb{Z}$–algebra $R := \mathbb{Z}[X]/(2X+1)$.

(a) Show that there is a ring isomorphism $R \simeq S^{-1}\mathbb{Z}$, where S is the multiplicatively closed subset of $\mathbb{Z}$ consisting of all the non-negative integral powers of 2, i.e. $S := \{1, 2, 2^2, 2^3, \dots\}$.

(b) Is R a flat $\mathbb{Z}$–module? Justify your answer.

(c) Is R a projective $\mathbb{Z}$–module? Justify your answer.

(d) Is R an injective $\mathbb{Z}$–module? Justify your answer.

Question 3.3. Let $R := \mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$, viewed as a subring of the field of complex numbers with the usual operations.

(a) Show that the group of units $R^\times$ of R is infinite.

(b) Show that the element 2 is irreducible in R.

(c) Show that the element 2 is not prime in R.

(d) Show that every element in R can be written as a product of irreducible elements.

(e) Is R a unique factorization domain? Justify your answer.

Hint. Recall that there is a multiplicative norm map $N : \mathbb{Z}[\sqrt{10}] \longrightarrow \mathbb{Z}$, given by $N(a + b\sqrt{10}) = a^2 - 10b^2$, for all a, $b \in \mathbb{Z}$.

4. Field Theory and Galois Theory

Question 4.1. Find the splitting fields and Galois groups of the following polynomials over $\mathbb{Q}$. You should state clearly results that you use.

(a) $(x^2 + 3)(x^3 - 5)$.

(b) $(x^2 - 3)(x^3 - 7)$.

(c) $x^{15} - 2$.

(d) $x^3 + 2x^2 + 1$.