Name: ____________________

PID: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	20	
Total:	80	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even if you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Suppose $p < q$ are two odd primes. Prove that a group of order p^2q is solvable.

2. (10 points) Suppose G is a finite group, $N \trianglelefteq G$ and P is a Sylow p-subgroup of N. Prove that $N_G(P) \cdot N = G$.

3. Suppose p is a prime number and the minimal polynomial of $g \in \mathrm{GL}_p(F)$ is $t^p - 1$.

   (a) (5 points) Find the Jordan form of g if $F = \mathbb{C}$.

(b) (5 points) Find the Jordan form of g if $F = \overline{\mathbb{F}}_p$ is an algebraic closure of the finite field $\mathbb{F}_p$.

4. (10 points) Suppose A is a PID, and M is a finitely generated A-module. Prove that M is projective if and only if M is free.

5. (10 points) Suppose A is a unital commutative ring and $\langle f_1, \ldots, f_n \rangle = A$ where $\langle f_1, \ldots, f_n \rangle$ is the ideal generated by f_i's. Suppose $M \subseteq N$ are two A-modules, and, for any i, we have $S_{f_i}^{-1} M = S_{f_i}^{-1} N$ where $S_{f_i} := \{1, f_i, f_i^2, \ldots\}$. Prove that $M = N$. (Hint: for $x \in N$ consider $\{a \in A | \ ax \in M\}$.)

6. (10 points) Suppose $f(x) \in \mathbb{F}_p[x]$ is an irreducible factor of $x^{p^n} - x$ where p is a prime number. Prove that $\deg f$ divides n.

7. Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial of degree $p+1$ where p is a prime. Let E be a splitting field of f over $\mathbb{Q}$. Suppose $[E : \mathbb{Q}] = p(p+1)$.

   (a) (8 points) Prove that for any zero $\alpha \in E$ of f, $E/\mathbb{Q}[\alpha]$ is a Galois extension and $\mathrm{Gal}(E/\mathbb{Q}[\alpha]) \simeq \mathbb{Z}/p\mathbb{Z}$.

(b) (12 points) Prove that there is $\beta \in E$ such that $\mathbb{Q}[\beta]/\mathbb{Q}$ is a Galois extension and $\mathrm{Gal}(\mathbb{Q}[\beta]/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$. (Hint: You can use whatever we have proved about groups of order $p(p+1)$.)

Good Luck!