ALGEBRA QUALIFYING EXAM, SPRING 2021

All problems are worth 15 points.

1. Classify up to isomorphism all groups G of order 56 with the property that all Sylow subgroups of G are cyclic. Write down a presentation for each group you find.

2. Let p be a prime and let H be a subgroup of the symmetric group S_p such that $|H| = p$.

(a). Show that the centralizer of H in S_p is H; that is, $C_{S_p}(H) = H$.

(b). Suppose that $H \subseteq N \subseteq S_p$ where N is a nilpotent group. Prove that $H = N$.

3. Let $f = x^6 - 5$. Let K be the splitting field of f over $F = \mathbb{Q}(\sqrt{5})$. Find the Galois group $\mathrm{Gal}(K/F)$ and show it is isomorphic to a familiar group.

4. Let K be a field with $|K| = 64$. Let $\mathbb{F}_2$ be the prime subfield of K. Let $G = \mathrm{Gal}(K/\mathbb{F}_2)$ act in the natural way on K, where for $\sigma \in G$ and $a \in K$ we have $\sigma \cdot a = \sigma(a)$.

(a). Describe the orbits of this action and calculate how many distinct orbits there are of each size.

(b). Calculate the number of $\alpha \in K$ such that $K = \mathbb{F}_2(\alpha)$.

(c). Calculate the number of $\beta \in K$ such that β generates the group $K^\times$.

5. For $i \geq 0$, calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q})$.

6. A is a matrix in $M_{12}(\mathbb{C})$. The characteristic polynomial of A is $(x+2)^3(x-1)^3(x+1)^6$; the minimal polynomial of A is $(x+2)^2(x-1)(x+1)^4$; A has exactly 3 invariant factors; and A has exactly 7 elementary divisors.

Show that there is exactly one similarity class of such matrices, and find an explicit matrix A in this similarity class.

7. Let $R = \mathbb{Z}[x]/(2x-1)$. Let $S = \{1, 2, 4, 8, \ldots\} \subseteq \mathbb{Z}$.

(a). Show that R is isomorphic as a ring to the localization $S^{-1}\mathbb{Z}$.

(b). Is R a free $\mathbb{Z}$-module?