UNIVERSITY OF CALIFORNIA, SAN DIEGO

Proof of CHSH Inequality and Tsirelson's Bound Using Same Method

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DEDICATION

To friends, family and my research group.

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Abstract of Honor Thesis

The CHSH inequality and Tsirelson's bound have traditionally been proven using different methods in the field of quantum mechanics. In this study, we employ a unified approach, utilizing multivariable calculus to derive and prove both the CHSH inequality and Tsirelson's bound. Proving both theorems using the same method is important as it offers a coherent and integrated perspective, enhancing our understanding of their interrelation and the underlying mathematical structure. This unified approach also provides deeper insights into the nature of quantum correlations compared to classical correlations. Furthermore, this method could potentially be applied to the research of the violation of Tsirelson's bound in future, highlighting the significance of multivariable calculus in the study of quantum mechanics.

Chapter 1

Proof of CHSH Inequality

The CHSH inequality is used to test the limits of correlations predicted by classical physics. It is formulated as:

$$|E(AB) + E(A'B) + E(AB') - E(A'B')| \leq 2$$

where $E(AB)$ represents the expectation value of measurements A and B.

1.1 Proof of CHSH Inequality by others[1]

In this section, I would provided a background information on how other people prove CHSH inequality.

For systems with two distant particles i and j, let A and a (B and b) be observables taking values -1 or 1. The correlation $C(A, B)$ is defined as:

$$C(A, B) = P_{AB}(1, 1) - P_{AB}(1, -1) - P_{AB}(-1, 1) + P_{AB}(-1, -1),$$

where $P_{AB}(1, -1)$ denotes the joint probability of $A = 1$ and $B = -1$. In any local-realistic theory, the absolute value of a combination of correlations is bound by 2:

$$|C(A, B) - mC(A, b) - nC(a, B) - mnC(a, b)| \leq 2,$$

where m and n can be either -1 or 1.

Derivation

In a local-realistic theory, the observables A, a, B, and b have predefined values v_A, v_a, v_B, and v_b, either -1 or 1. Thus, the combination can be calculated as:

$$v_B(v_A - nv_a) - mv_b(v_A + nv_a),$$

which is either -2 or 2. Therefore, the absolute value of the averages is bound by 2.

Cirel'son Bound

For a two-particle system in a quantum pure state $|\psi\rangle$, the quantum correlation of A and B is:

$$C_Q(A, B) = \langle\psi|\hat{A}\hat{B}|\psi\rangle,$$

where $\hat{A}$ and $\hat{B}$ are the self-adjoint operators representing observables A and B. Cirel'son demonstrated that for a two-particle system, the combination of quantum correlations is bound by $2\sqrt{2}$:

$$|C_Q(A, B) - mC_Q(A, b) - nC_Q(a, B) - mnC_Q(a, b)| \leq 2\sqrt{2}.$$

1.2 Proof Using Multivariable Calculus

Suppose matrices A and B are define as following:

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \quad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

The tensor product of Matrices A and B are $A \otimes B$ is given by:

$$A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{pmatrix} = \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{pmatrix}$$

Since we are considering for classical circuits, The state we choose is $|\psi\rangle$ defined as:

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$$

To derive and prove the CHSH inequality using multivariable calculus, we consider the setup involving markov matrices which also known as stochastic matrices A and B. Thus we can define the matrix as following:

For matrix A:

$$A = [a_{ij}] = \begin{bmatrix} a & a' \\ 1-a & 1-a' \end{bmatrix} where$$

$$\sum_i a_{ij} = 1 \quad \text{for all columns } j$$

$$a_{ij} \geq 0 \quad \text{for all entries } a_{ij}$$

For matrix B:

$$B = [b_{ij}] = \begin{bmatrix} b & b' \\ 1-b & 1-b' \end{bmatrix} where$$

$$\sum_i b_{ij} = 1 \quad \text{for all columns } j$$

$$b_{ij} \geq 0 \quad \text{for all entries } b_{ij}$$

Similarly, we can define markov matrix A' with variable c,c' and matrix B' with variable d,d'.

We first compute the expectation values using tensor product:

$$E(AB) = \langle\psi|A \otimes B|\psi\rangle = (a - a')(b - b')$$

$$E(A'B) = (c - c')(b - b')$$

$$E(AB') = (a - a')(d - d')$$

$$E(A'B') = (c - c')(d - d')$$

We then define the CHSH function F as:

$$\begin{aligned} F &= E(AB) + E(A'B) + E(AB') - E(A'B') \\ &= (a-a')(b-b') + (c-c')(b-b') + (a-a')(d-d') - (c-c')(d-d') \end{aligned}$$

1.2.1 Derivatives Calculation

To find the max and min, we need to take 1st and 2nd deratives of function F. We take the partial derivatives of function F respect to a,a',b,b',c,c',d,d':

$$\frac{\partial F}{\partial a} = (b-b') + (d-d'), \quad \frac{\partial F}{\partial a'} = -(b-b') - (d-d')$$

$$\frac{\partial F}{\partial b} = (a-a') + (c-c'), \quad \frac{\partial F}{\partial b'} = -(a-a') - (c-c')$$

$$\frac{\partial F}{\partial c} = (b-b') - (d-d'), \quad \frac{\partial F}{\partial c'} = -(b-b') + (d-d')$$

$$\frac{\partial F}{\partial d} = (a-a') - (c-c'), \quad \frac{\partial F}{\partial d'} = -(a-a') + (c-c')$$

Then we take second derivatives of and compute the Hessian matrix H:

$$H = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

The eigenvalues of the Hessian matrix are:

$$\lambda_1 = -2.828, \lambda_2 \approx 0, \lambda_3 = 2.828, \lambda_4 = -2.828, \lambda_5 \approx 0, \lambda_6 = 2.828, \lambda_7 = 0, \lambda_8 = 0$$

Since there is both negative, positive and 0 eigenvalues, we can conclude that the critical points inside the boundary are saddle point. By checking the boundary point for a,b,c,d,a',b',c',d' be either 0 or 1, we find that

$$F_{\max} = 2$$

$$F_{\min} = -2$$

at the boundary point.

For example, when a=d=d'=c=b=1,a'=b'=c'=0, we have f=1+1=2. When a'=d=d'=c'=b=1,a=b'=c=0, we have f=(-1)+(-1)=-2.

Based on the max and min, we would able to get the inequality,

$$|F| \leq 2$$

Thus, the CHSH inequality is proved using multivariable calculus.

Chapter 2

Proof of Tsirelson'Bound

Tsirelson's bound is a fundamental result in quantum mechanics that sets the upper limit on the violation of the CHSH inequality in quantum systems. It provides a quantum mechanical limit to the correlations that can be observed, which surpasses the classical limit set by the CHSH inequality. Tsirelson's bound is given by:

$$|E(AB) + E(A'B) + E(AB') - E(A'B')| \leq 2\sqrt{2}$$

2.1 Proof of Tsirelson'Bound by others[1]

In this section, I would provided a background information on how other people prove Tsirelson'Bound inequality. Using the same notation and definitaion from 1.1, Consider the operator:

$$\hat{C} = \hat{A}\hat{B} - m\hat{A}\hat{b} - n\hat{a}\hat{B} - mn\hat{a}\hat{b}.$$

If $\hat{A}^2 = \hat{a}^2 = \hat{B}^2 = \hat{b}^2 = I$ (identity operator), then:

$$\hat{C}^2 = 4I - mn[\hat{A}, \hat{a}][\hat{B}, \hat{b}].$$

Since for all bounded operators $\hat{F}$ and $\hat{G}$,

$$||[\hat{F}, \hat{G}]|| \leq ||\hat{F}\hat{G}|| + ||\hat{G}\hat{F}|| \leq 2||\hat{F}||||\hat{G}||,$$

then $||\hat{C}^2|| \leq 8$, or $||\hat{C}|| \leq 2\sqrt{2}$. This shows that the absolute value of the combination of quantum correlations is bound by $2\sqrt{2}$.

2.2 Proof Using Multivariable Calculus

To prove Tsirelson's bound, we use rotation matrices for the measurements A, B, A', and B'. The rotation matrix $R(\theta)$ is defined as:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Specifically, the measurement matrices are defined as follows:

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$A' = \begin{pmatrix} \cos\alpha' & -\sin\alpha' \\ \sin\alpha' & \cos\alpha' \end{pmatrix}$$

$$B' = \begin{pmatrix} \cos\beta' & -\sin\beta' \\ \sin\beta' & \cos\beta' \end{pmatrix}$$

These matrices represent rotations by angles α, β, α', and β' respectively, and are used to describe the measurement settings for the quantum system.

We consider the Bell state $|\Phi^+\rangle$, which is defined as:

$$|\Phi^+\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$$

This state is maximally entangled and is used to exhibit the quantum correlations that we seek to analyze.

The expectation values for the measurements are calculated using the Bell state and the rotation matrices. Given that

$$E(AB) = \langle\psi|A \otimes B|\psi\rangle$$

We compute the expectation values:

$$E(AB) = \cos(2(\alpha - \beta))$$

$$E(A'B) = \cos(2(\alpha' - \beta))$$

$$E(AB') = \cos(2(\alpha - \beta'))$$

$$E(A'B') = \cos(2(\alpha' - \beta'))$$

We define the function F that captures the combination of expectation values involved in Tsirelson's bound:

$$F = E(AB) + E(A'B) + E(AB') - E(A'B')$$

Expanding the expectation values, we have:

$$F = \cos(2(\alpha - \beta)) + \cos(2(\alpha' - \beta)) + \cos(2(\alpha - \beta')) - \cos(2(\alpha' - \beta'))$$

2.2.1 Derivatives Calculation

To find the extrema of the function F, we compute its partial derivatives with respect to the angles α, α', β, and β':

$$\frac{\partial F}{\partial \alpha} = -2\sin(2(\alpha - \beta)) + 2\sin(2(\alpha - \beta'))$$

$$\frac{\partial F}{\partial \alpha'} = -2\sin(2(\alpha' - \beta)) - 2\sin(2(\alpha' - \beta'))$$

$$\frac{\partial F}{\partial \beta} = 2\sin(2(\alpha - \beta)) - 2\sin(2(\alpha' - \beta))$$

$$\frac{\partial F}{\partial \beta'} = 2\sin(2(\alpha - \beta')) + 2\sin(2(\alpha' - \beta'))$$

To find the critical points, we set the partial derivatives to zero:

$$\frac{\partial F}{\partial \alpha} = -2\sin(2(\alpha - \beta)) + 2\sin(2(\alpha - \beta')) = 0$$

$$\frac{\partial F}{\partial \alpha'} = -2\sin(2(\alpha' - \beta)) - 2\sin(2(\alpha' - \beta')) = 0$$

$$\frac{\partial F}{\partial \beta} = 2\sin(2(\alpha - \beta)) - 2\sin(2(\alpha' - \beta)) = 0$$

$$\frac{\partial F}{\partial \beta'} = 2\sin(2(\alpha - \beta')) + 2\sin(2(\alpha' - \beta')) = 0$$

By solving these equations, we can find the critical points. Specifically, we get:

$$\sin(2(\alpha - \beta)) = \sin(2(\alpha - \beta')) = \sin(2(\alpha' - \beta)) = -\sin(2(\alpha' - \beta'))$$

$$\sin(2(\alpha - \beta)) = \sin(2(\alpha' - \beta)) = \sin(2(\alpha - \beta')) = -\sin(2(\alpha' - \beta'))$$

Since we are only caring the difference between two variable when calculate function F, we can set $\beta = 0$. We get $\alpha - \beta = \alpha$, thus we compute critical points are:

$$\alpha - \beta' = -\alpha + k\pi \quad \text{or} \quad \alpha - \beta' = \alpha + l\pi$$

$$\alpha' - \beta = \beta' - \alpha + k_1\pi \quad \text{or} \quad \alpha' - \beta = \alpha + l_1\pi$$

$$\alpha' - \beta' = -\alpha + k_2\pi \quad \text{or} \quad \alpha' - \beta' = \alpha + l_2\pi$$

where k, k_1, k_2, l, l_1, l_2 are integers.

Then we take second derivatives of and compute the Hessian matrix: The Hessian matrix H is given by:

$$H = \begin{pmatrix} -4\cos(2(\alpha-\beta)) + 4\cos(2(\alpha-\beta')) & 0 & 4\cos(2(\alpha-\beta)) & -4\cos(2(\alpha-\beta')) \\ 0 & -4\cos(2(\alpha'-\beta)) - 4\cos(2(\alpha'-\beta')) & -4\cos(2(\alpha'-\beta)) & -4\cos(2(\alpha'-\beta')) \\ 4\cos(2(\alpha-\beta)) & -4\cos(2(\alpha'-\beta)) & -4\cos(2(\alpha-\beta)) + 4\cos(2(\alpha'-\beta)) & 0 \\ -4\cos(2(\alpha-\beta')) & -4\cos(2(\alpha'-\beta')) & 0 & -4\cos(2(\alpha-\beta')) + 4\cos(2(\alpha'-\beta')) \end{pmatrix}$$

After test the critical points by Hessian matrix and plug back to F, we find:

$$F_{\max} = 2\sqrt{2}$$

$$F_{\min} = -2\sqrt{2}$$

Based on the max and min, we would able to get the inequality,

$$|F| \leq 2\sqrt{2}$$

Thus, the Tsirelson'Bound is proved using multivariable calculus.

Chapter 3

Conclusion

In this thesis, we have explored and proven two significant inequalities in the field of quantum mechanics: the CHSH inequality and Tsirelson's bound. Both inequalities play a important role in understanding the limits of correlations that can be observed in quantum systems versus classical systems. Proving both inequality using multivariable calculus help us have a integrate idea about the two inequality. while study future violation of Tsirelson's Bound, Multivariable calculus maybe an effective method to use since it's able to prove for both CHSH inequality and Tsirelson's bound.

Bibliography

[1] Adán Cabello. "Violating Bell's Inequality Beyond Cirel'son's Bound". In: *Physical Review Letters* 88.6 (Jan. 2002). ISSN: 1079-7114. DOI: `10.1103/physrevlett.88.060403`. URL: `http://dx.doi.org/10.1103/PhysRevLett.88.060403`.