Character Varieties and Orbit Stability mod p^k

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1 Representation and Character Varieties

For n a positive integer and Γ a group, let Hom(Γ,SL_n) denote the functor which sends a unital commutative ring R to the set of group homomorphisms Hom(Γ,$\mathrm{SL}_n(R)$). We call this the SL_n*-representation variety* of the group Γ.

We shall define the *universal representation algebra* A(Γ,SL_n) as follows:

Consider a set of indeterminates $\{a_{ij}(g)\}_{1\leq i,j\leq n,\, g\in\Gamma}$.

For each $g \in \Gamma$ define the $n \times n$ matrix $\sigma(g) := (a_{ij}(g))_{1\leq i,j\leq n}$.

Define the ideal

$$I := \langle a_{ij}(e)-\delta_{ij},\ a_{ij}(g_1g_2)-\sum_{k=1}^{n} a_{ik}(g_1)a_{kj}(g_2),\ \det(\sigma(g))-1 \mid g_1, g_2, g \in \Gamma, 1 \leq i, j \leq n\rangle.$$

Then

$$A(\Gamma, \mathrm{SL}_n) := \frac{\mathbb{Z}[a_{ij}(g) \mid g \in \Gamma, 1 \leq i, j \leq n]}{I}.$$

Now define the *universal representation of* Γ *in* SL_n $\sigma : \Gamma \to \mathrm{SL}_n(A(\Gamma, \mathrm{SL}_n))$ by

$$\sigma(g) := (a_{ij}(g))_{1\leq i,j\leq n},$$

where now we are viewing the a_{ij}'s as elements of $A(\Gamma, \mathrm{SL}_n)$.

Lemma 1. $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ *is a representable functor represented by* $A(\Gamma, \mathrm{SL}_n)$.

(See Section 1.1 of [13] for further explanation of this result.)

By an abuse of notation, we shall use Hom(Γ,SL_n) and the term "representation variety" to refer to the affine scheme over $\mathbb{Z}$ given by $\mathrm{Spec}(A(\Gamma, \mathrm{SL}_n))$.

Now we shall define another ring, the *universal character ring of representations of* Γ *in* SL_2, by

$$R(\Gamma, \mathrm{SL}_2) := \frac{\mathbb{Z}[t_g \mid g \in \Gamma]}{\langle t_e - 2, t_{g_1}t_{g_2} - t_{g_1g_2} - t_{g_1^{-1}g_2} \mid g_1, g_2 \in \Gamma\rangle},$$

where the relations we are modding out by are referred to as the *Fricke identities*.

We shall define the SL_2-*character variety of* Γ to be the affine scheme given by taking Spec of the universal character ring, that is,

$$\mathrm{Ch}(\Gamma, \mathrm{SL}_2) := \mathrm{Spec}(R(\Gamma, \mathrm{SL}_2)).$$

Define a ring homomorphism $\Phi : R(\Gamma, \mathrm{SL}_2) \to A(\Gamma, \mathrm{SL}_2)$ by

$$\Phi(t_g) := tr(\sigma(g)).$$

It follows precisely from the Fricke identities that Φ is well-defined.

Recall that every ring homomorphism $B \to A$ induces a morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$, and conversely, every morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is induced by some ring homomorphism $B \to A$ (see Proposition 2.3 of [10]).

Thus, Φ induces a morphism $\pi_\Gamma : \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to \mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ from the SL_2-representation variety of Γ to the SL_2-character variety of Γ, which we call the *invariant morphism*.

It is a result of Robert Horowitz (see Theorem 3.1 of [11]) that if Γ is a finitely generated group with a linearly ordered generating set Ω, then $R(\Gamma, \mathrm{SL}_2)$ is a finitely generated ring, with a generating set over $\mathbb{Z}$ given by

$$\{t_\omega \mid n \in \mathbb{N},\ \omega = g_1 g_2 \cdots g_n,\ g_1, ..., g_n \in \Omega,\ g_1 < g_2 < \cdots < g_n\}.$$

$R(\Gamma, \mathrm{SL}_2)$ being a finitely generated ring over $\mathbb{Z}$ with this generating set means that there exist polynomials

$$p_1, ..., p_s \in \mathbb{Z}[t_\omega \mid n \in \mathbb{N},\ \omega = g_1 g_2 \cdots g_n,\ g_1, ..., g_n \in \Omega,\ g_1 < g_2 < \cdots < g_n]$$

such that

$$R(\Gamma, \mathrm{SL}_2) \cong \frac{\mathbb{Z}[t_\omega \mid n \in \mathbb{N},\ \omega = g_1 g_2 \cdots g_n,\ g_1, ..., g_n \in \Omega,\ g_1 < g_2 < \cdots < g_n]}{\langle p_1, ..., p_s \rangle}.$$

If we let

$$\ell := |\{\omega \in \Gamma \mid n \in \mathbb{N},\ \omega = g_1 g_2 \cdots g_n,\ g_1, ..., g_n \in \Omega,\ g_1 < g_2 < \cdots < g_n\}|,$$

then for a ring R, the set of R-points of the SL_2-character variety of Γ is thus given by

$$\mathrm{Ch}(\Gamma, \mathrm{SL}_2)(R) := \{\overrightarrow{r} \in R^\ell \mid p_i(\overrightarrow{r}) = 0 \ \ \forall\, 1 \leq i \leq s\}.$$

Hence we have a morphism $\pi_\Gamma(R) : \mathrm{Hom}(\Gamma, \mathrm{SL}_2(R)) \to \mathrm{Ch}(\Gamma, \mathrm{SL}_2)(R)$ given by

$$\pi_\Gamma(R)(\rho) := (tr(\rho(\omega)) \mid n \in \mathbb{N}, \omega = g_1 g_2 \cdots g_n, g_1, ..., g_n \in \Omega, g_1 < g_2 < \cdots < g_n).$$

An alternative way of thinking about $\pi_\Gamma(R)$ is by viewing t_g, for each $g \in \Gamma$, as a regular function of $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ such that for every $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2(R))$,

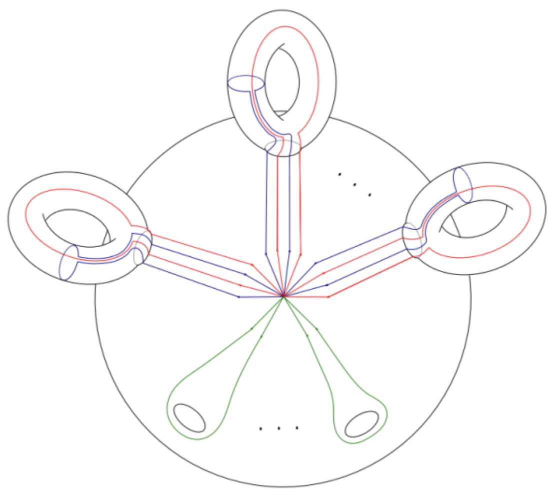
$$t_g(\pi_\Gamma(R)(\rho)) = tr(\rho(g)).$$

Moving forward, we will often simply write π_Γ instead of $\pi_\Gamma(R)$ if the ring R is clear from context.

For the remainder of this work, we shall be concerned with representation varieties and character varieties of surface groups. By surface groups, we mean fundamental groups of compact, connected, orientable surfaces with finitely many punctures. (Of course, after puncturing, the surface is no longer compact.) Let $\Sigma_{g,n}$ denote such a surface with genus g and n punctures. Then the fundamental group of $\Sigma_{g,n}$ with respect to some arbitrary basepoint x_0, which we shall denote by $\Pi_{g,n}$, is given by the presentation:

$$\Pi_{g,n} := \pi_1(\Sigma_{g,n}, x_0) \cong \langle a_1, b_1, ..., a_g, b_g, c_1, ..., c_n \mid [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] c_1 \cdots c_n \rangle,$$

where a_i's, b_i's, and c_i's correspond to the homotopy classes of the blue, red, and green loops in the following diagram, respectively:



For convenience, we introduce the following notation:

$$\mathrm{Rep}_{g,n} := \mathrm{Hom}(\Pi_{g,n}, \mathrm{SL}_2)$$

$$\mathrm{Ch}_{g,n} := \mathrm{Ch}(\Pi_{g,n}, \mathrm{SL}_2).$$

$$\pi_{g,n} := \pi_{\Pi_{g,n}}$$

2 Examples

Let F_n denote the free group of rank n. Observe that the representation and character variety corresponding to a given surface $\Sigma_{g,n}$ depend only on the fundamental group $\Pi_{g,n}$. For the three-punctured sphere and the one-punctured torus, we get a neat result, for which the details can be found in Section 6.2 of [2]:

Lemma 2. $R(F_2, \mathrm{SL}_2) \cong \mathbb{Z}[x_1, x_2, x_3]$.

Notice that $\Pi_{0,3} \cong F_2$ and $\Pi_{1,1} \cong F_2$. Therefore, the lemma demonstrates that $\mathrm{Ch}_{0,3} \cong \mathbb{A}^3$ and $\mathrm{Ch}_{1,1} \cong \mathbb{A}^3$.

The case of the four-holed sphere and the two-holed torus is less simple:

Lemma 3. *Define the polynomial* $p \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ *by*

$$p := \sum_{j=1}^{7} x_j^2 - (x_1x_2x_4 + x_2x_3x_6 + x_1x_3x_5 + x_3x_4x_7 + x_1x_6x_7 + x_2x_5x_7) + x_4x_5x_6 + x_1x_2x_3x_7 - 4.$$

We have that $R(F_3, \mathrm{SL}_2) \cong \frac{\mathbb{Z}[x_1,x_2,x_3,x_4,x_5,x_6,x_7]}{\langle p \rangle}$.

For details on this case see Section 1.5 of [4].

Notice this time that $\Pi_{0,4} \cong F_3$ and $\Pi_{1,2} \cong F_3$. Thus the lemma effectively gives us $\mathrm{Ch}_{0,4}$ and $\mathrm{Ch}_{1,2}$. This example demonstrates that $\mathrm{Ch}_{g,n}$ is not always affine and may not be easy to understand in general.

3 Action of Mapping Class Group

Definition 1. The **mapping class group** of a surface $\Sigma_{g,n}$, denoted $\mathrm{MCG}(\Sigma_{g,n})$, is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ which restrict to the identity on the boundary $\partial\Sigma_{g,n}$.

Definition 2. The **pure mapping class group** of a surface $\Sigma_{g,n}$, denoted $\Gamma_{g,n}$, is the subgroup of $\mathrm{MCG}(\Sigma_{g,n})$ consisting of elements which fix each puncture individually.

I now aim to describe the action of $\mathrm{MCG}(\Sigma_{g,n})$ on $\mathrm{Ch}_{g,n}$.

Lemma 4. *If* Γ *is a group and* $g_1, g_2 \in \Gamma$, *then* $t_{g_1g_2g_1^{-1}} = t_{g_2}$, *viewed as elements of* $R(\Gamma, \mathrm{SL}_2)$.

Let $[\varphi] \in \mathrm{MCG}(\Sigma_{g,n})$, where φ is an orientation-preserving homeomorphism of $\Sigma_{g,n}$ fixing the boundary pointwise. Recall $\Pi_{g,n} = \pi_1(\Sigma_{g,n}, x_0)$, where x_0 was an arbitrarily chosen basepoint. For a choice of a path $\rho : I \to \Sigma_{g,n}$ from x_0 to $\varphi(x_0)$, we can define a homomorphism $\varphi_* : \Pi_{g,n} \to \Pi_{g,n}$ by

$$\varphi_*([\gamma]) := [\rho \cdot (\varphi \circ \gamma) \cdot \rho^{-1}].$$

Since φ is a homeomorphism and is thus invertible, it follows that φ_* is invertible and is thus an automorphism. Choosing a different path ρ' from x_0 to $\varphi(x_0)$ results in a different automorphism φ'_*, which is equivalent to φ_* composed with an inner automorphism. Thus, what we have described defines a group homomorphism from $\mathrm{MCG}(\Sigma_{g,n})$ to $\mathrm{Out}(\Pi_{g,n})$, where an isotopy class $[\varphi]$ gets sent to the automorphism class $[\varphi_*]$. For $[\varphi] \in \mathrm{MCG}(\Sigma_{g,n})$, define a ring homomorphism

$$\tilde{\varphi} : \mathbb{Z}[t_\gamma \mid \gamma \in \Pi_{g,n}] \to R(\Pi_{g,n}, \mathrm{SL}_2)$$

by

$$\tilde{\varphi}(t_\gamma) := t_{\varphi_*(\gamma)},$$

where φ_* is in $[\varphi_*]$, the $\mathrm{Out}(\Pi_{g,n})$ element associated to $[\varphi]$, and the choice of $\varphi_* \in [\varphi_*]$ doesn't matter by Lemma 4. Notice firstly that $\tilde{\varphi}$ is surjective. Also,

$$\ker \varphi_* = \langle t_e - 2,\ t_{\gamma_1} t_{\gamma_2} - t_{\gamma_1\gamma_2} - t_{\gamma_1^{-1}\gamma_2} \mid \gamma_1, \gamma_2 \in \Pi_{g,n} \rangle.$$

Therefore $\tilde{\varphi}$ induces a ring automorphism $\bar{\tilde{\varphi}}$ of $R(\Pi_{g,n}, \mathrm{SL}_2)$, which in turn induces a scheme automorphism $\hat{\varphi}$ of $\mathrm{Ch}_{g,n}$. It follows that $\mathrm{MCG}(\Sigma_{g,n})$ acts on $\mathrm{Ch}_{g,n}$, where an element $[\varphi]$ is associated to the scheme automorphism $\hat{\varphi}$. This action restricts to an action of $\Gamma_{g,n}$ on $\mathrm{Ch}_{g,n}$.

4 Result of Golsefidy and Tamam

In this section we will develop the terminology for and then state the theorem from [14] upon which this thesis is meant to build upon.

Definition 3. For $\gamma, \gamma' \in \Pi_{g,n}$, the *discriminant* $\Delta(\gamma, \gamma') \in R(\Pi_{g,n}, \mathrm{SL}_2)$ is defined by

$$\Delta(\gamma, \gamma') := t_{[\gamma, \gamma']} - 2.$$

Manipulation using the Fricke identities gives that for $\gamma, \gamma' \in \Pi_{g,n}$,

$$\Delta(\gamma, \gamma') = t_\gamma^2 + t_{\gamma'}^2 + t_{\gamma\gamma'}^2 - t_\gamma t_{\gamma'} t_{\gamma\gamma'} - 4.$$

Definition 4. The *discriminant subvariety* $D_{g,n}$ of $\mathrm{Ch}_{g,n}$ is the subvariety given by the ideal

$$\langle \Delta(\gamma, \gamma') \mid \gamma, \gamma' \in \Pi_{g,n} \rangle \trianglelefteq R(\Pi_{g,n}, \mathrm{SL}_2).$$

Definition 5. Define Zariski-open subschemes $\mathrm{Ch}_{g,n}^\times$ and $\mathrm{Rep}_{g,n}^\times$ of $\mathrm{Ch}_{g,n}$ and $\mathrm{Rep}_{g,n}$, respectively, by

$$\mathrm{Ch}_{g,n}^\times := \mathrm{Ch}_{g,n} \setminus D_{g,n}$$
$$\mathrm{Rep}_{g,n}^\times := \mathrm{Rep}_{g,n} \setminus \pi_{g,n}^{-1}(D_{g,n}).$$

The following result of Golsefidy and Tamam is directly from Section 2.2 of [14]:

Lemma 5. *Let G be a group isomorphic to either $\mathrm{SL}_2(\mathbb{F}_3)$, a double cover of S_4, or $\mathrm{SL}_2(\mathbb{F}_5)$. Then there exists a closed subscheme $\underline{F}_{g,n;G}$ of $\mathrm{Ch}^{\times}_{g,n}$ such that for every algebraically closed field F of characteristic either zero or more than 5,*

$$\pi^{-1}_{g,n}(\underline{F}_{g,n;G}(F)) = \{\rho \in \mathrm{Rep}^{\times}_{g,n}(F) \mid \mathrm{Im}(\rho) \cong G\}.$$

This lemma serves as a definition of $\underline{F}_{g,n;G}$.

Definition 6. For $\mathcal{R}$ a subset of a free group on $2g+n$ generators, let $\mathrm{Rep}^{\times}_{g,n,\mathcal{R}}$ denote the closed subscheme of $\mathrm{Rep}^{\times}_{g,n}$ such that for every unital commutative ring R,

$\rho \in \mathrm{Rep}^{\times}_{g,n,\mathcal{R}}(R) \iff \rho \in \mathrm{Rep}^{\times}_{g,n}(R)$ and $\forall w \in \mathcal{R}$, $w(\rho) = 1$.

Similarly, let $\mathrm{Ch}^{\times}_{g,n;\mathcal{R}}$ denote the closed subscheme of $\mathrm{Ch}^{\times}_{g,n}$ such that for every unital commutative ring R,

$x \in \mathrm{Ch}^{\times}_{g,n;\mathcal{R}}(R) \iff x \in \mathrm{Ch}^{\times}_{g,n}(R)$ and $\forall w \in \mathcal{R}$, $\forall s_1, s_2, s_3 \in S$, $t_{s_1 s_2 s_3}(x) = t_{w s_1 s_2 s_3}(x)$,

where $S := \{1, a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_n\}$.

It follows from this definition that $\pi_{g,n}$ induces a well-defined morphism from $\mathrm{Rep}^{\times}_{g,n,\mathcal{R}}$ to $\mathrm{Ch}^{\times}_{g,n;\mathcal{R}}$.

Definition 7. For every subset I of

$$\{a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_n\},$$

let $\mathcal{R}_I$ denote the subset of the free group on $2g+n$ generators generated by I. Then define

$$\mathrm{Rep}^{\bullet}_{g,n} := \mathrm{Rep}^{\times}_{g,n} \setminus (\bigcup_I \mathrm{Rep}^{\times}_{g,n,\mathcal{R}_I} \cup \bigcup_G \pi^{-1}_{g,n}(\underline{F}_{g,n;G})),$$

where I ranges over all subsets of $\{a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_n\}$ and G ranges over $\mathrm{SL}_2(\mathbb{F}_3)$, a double cover of S_4, and $\mathrm{SL}_2(\mathbb{F}_5)$. Similarly, define

$$\mathrm{Ch}^{\bullet}_{g,n} := \mathrm{Ch}^{\times}_{g,n} \setminus (\bigcup_I \mathrm{Ch}^{\times}_{g,n;\mathcal{R}_I} \cup \bigcup_G \underline{F}_{g,n;G}).$$

Now let I be a subset of $\{1, ..., n\}$, and let $\epsilon := (\epsilon_i)_{i \in I}$ be a collection of signs (± 1). Golsefidy and Tamam define a closed subscheme $\mathrm{Ch}^{\times}_{g,n;\epsilon}$ of $\mathrm{Ch}^{\times}_{g,n}$ with the following property (see Section 3.1, Lemma 23 of [14]):

For any unital commutative ring R and element $x \in \mathrm{Ch}^{\times}_{g,n}(R)$, $x \in \mathrm{Ch}^{\times}_{g,n;\epsilon}(R)$ if and only if for every ring extension A of R and for every $\rho \in \mathrm{Rep}^{\times}_{g,n}(A)$ such that $\pi_{g,n}(\rho) = x$ (we call such a representation a *lift* of x) we have that $\rho(c_i) = \epsilon_i 1$ for all $i \in I$.

Now further let R be a unital commutative ring and let $\mathbf{k} := (k_i)_{i\in\{1,...,n\}\setminus I}$ be a collection of elements of R. Golsefidy and Tamam define a subscheme $\mathrm{Ch}^{\times}_{g,n;\epsilon,\mathbf{k}}$ of $\mathrm{Ch}^{\times}_{g,n;\epsilon} \times_{\mathbb{Z}} R$ given by the equation

$$(t_{c_i}(x))_{i\in\{1,...,n\}\setminus I} = \mathbf{k}.$$

That is, for B another unital commutative ring and $x \in \mathrm{Ch}^{\times}_{g,n;\epsilon,\mathbf{k}}(B)$, we require that for every lift ρ of x and every $i \in \{1, ..., n\}\setminus I$,

$$tr(\rho(c_i)) = k_i.$$

$\mathrm{Ch}^{\times}_{g,n;\epsilon,\mathbf{k}}$ is referred to as a *modified relative character variety* of $\Sigma_{g,n}$.

Finally, Golsefidy and Tamam define an open subscheme $\mathrm{C}_{g,n;\epsilon,\mathbf{k}}$ of $\mathrm{Ch}^{\times}_{g,n;\epsilon,\mathbf{k}}$ by requiring that for B any unital commutative ring, the number

$$|\{i \in \{1, ..., n\} \mid \rho(c_i) \neq \pm 1\}|$$

remains constant as x ranges over $\mathrm{Ch}^{\times}_{g,n;\epsilon,\mathbf{k}}(B)$ and ρ ranges over all lifts of x.

We are almost ready to state the result.

For $k \in \mathbb{Z}_{\geq 1}$, let

$$\mathrm{C}^{\bullet}_{g,n+m;\epsilon,\mathbf{k}}(\mathbb{Z}/p^k\mathbb{Z}) := \mathrm{C}_{g,n+m;\epsilon,\mathbf{k}}(\mathbb{Z}/p^k\mathbb{Z}) \cap \mathrm{Ch}^{\bullet}_{g,n+m}(\mathbb{Z}/p^k\mathbb{Z}),$$

where $n+m$ indicates that we are working with $\Sigma_{g,n+m}$ and $|I| = m$. Let $\overline{N}_{g,n;\epsilon,\mathbf{k}}(k)$ denote the number of $\Gamma_{g,n+m}$-orbits in $\mathrm{C}^{\bullet}_{g,n+m;\epsilon,\mathbf{k}}(\mathbb{Z}/p^k\mathbb{Z})$. Then the following is due to Golsefidy and Tamam (see Section 7.4, Theorem 98 in [14]):

Theorem 6. *Suppose p is a prime, g is a positive integer, n is a non-negative integer, $\boldsymbol{k} \in \mathbb{Z}_p^n$, and $\epsilon := (\epsilon_i)_i \in \{\pm 1\}^m$. Suppose one of the following conditions hold:*

1. *$g \geq 3$.*
2. *$g = 2$ and either $n > 0$ or $\prod_{i=1}^{m} \epsilon_i \neq -1$.*
3. *$g = 1$ and $n \neq 2$.*

Then there exists a positive integer $k_0 := k_0(g, n, \epsilon, \boldsymbol{k}, p)$ and a real number $c_0 := c_0(g, n, \epsilon, \boldsymbol{k}, p) \geq 1$ such that for all $k \geq k_0$, the following statements hold:

1. *$\overline{N}_{g,n;\epsilon,\boldsymbol{k}}(k) = \overline{N}_{g,n;\epsilon,\boldsymbol{k}}(k_0)$*
2. *For every $x \in \mathrm{C}^{\bullet}_{g,n+m;\epsilon,\boldsymbol{k}}(\mathbb{Z}/p^k\mathbb{Z})$,*

$$c_0^{-1} p^{dk} \leq |\Gamma_{g,n+m} \cdot x| \leq c_0 p^{dk},$$

where $d = 2(3g + n - 3)$.

We are concerned particularly with the first of the two statements. The result gives the existence of a number k_0 such that the number of $\Gamma_{g,n+m}$-orbits in $\mathrm{C}^{\bullet}_{g,n+m;\epsilon,\mathbf{k}}(\mathbb{Z}/p^k\mathbb{Z})$ is constant for increasing $k \geq k_0$. However, the method of proof does not provide any upper bounds on k_0. Our goal is to show that in specific cases, that is for explicit choices of p, g, n, $\mathbf{k}$, and ϵ, we have that k_0 is actually small, for instance $k_0 = 1$ or $k_0 = 2$. In a sense, this would demonstrate that in certain cases, the number of $\Gamma_{g,n+m}$-orbits in $\mathrm{C}^{\bullet}_{g,n+m;\epsilon,\mathbf{k}}(\mathbb{Z}/p^k\mathbb{Z})$ "stabilizes quickly" with respect to k. The hope is that this would shed light on the general behavior of k_0 with respect to the different parameters.

5 Pure mapping class group is finitely generated

Definition 8. A **closed curve** in a surface $\Sigma_{g,n}$ is defined to be a continuous map $S^1 \to \Sigma_{g,n}$. We say that a closed curve is **simple** if the corresponding map $S^1 \to \Sigma_{g,n}$ is injective.

Often when we refer to a closed curve, we are really referring to the image of the associated map.

We now define the notion of a Dehn twist, following closely to the exposition given in Chapter 3 of [3].

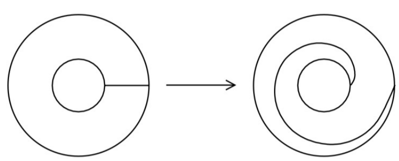
Consider the annulus $A = S^1 \times [0,1]$. We orient A by embedding it in the polar coordinate plane via the map

$$(\theta, t) \mapsto (\theta, t+1)$$

and giving it the orientation induced by the standard orientation of the plane. Now define the **twist map** $T : A \to A$ by

$$T(\theta, t) := (\theta + 2\pi t, t).$$

It may be helpful to see what T does to the set $\{(0,t) \mid t \in [0,1]\} \subseteq A$:



Notice that T is an orientation-preserving homeomorphism of A which fixes ∂A pointwise.

Remark. We could have defined T by $(\theta, t) \mapsto (\theta - 2\pi t, t)$. This would be a "right" twist, while our definition above is a "left" twist.

Let α be a simple closed curve in $\Sigma_{g,n}$. Let N be a regular neighborhood of α, and let ϕ be an orientation-preserving homeomorphism $A \to N$. Then the

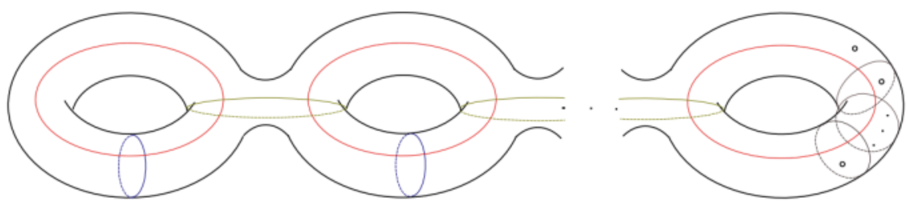
Dehn twist about α is the homeomorphism $T_\alpha : \Sigma_{g,n} \to \Sigma_{g,n}$ defined by

$$T_\alpha(x) := \begin{cases} \phi \circ T \circ \phi^{-1}(x) & x \in N \\ x & x \in \Sigma_{g,n} \backslash N \end{cases}$$

Observe that T_α always fixes $\partial\Sigma_{g,n}$. We see that T_α itself depends on the choice of regular neighborhood N and homeomorphism ϕ. However, via the theory of regular neighborhoods, the isotopy class of T_α does not depend on these choices. Furthermore, it also doesn't depend on the choice of the simple closed curve within the isotopy class of α. So if we let a denote the isotopy class of α, then T_a is a well-defined element of $\mathrm{MCG}(\Sigma_{g,n})$, which we shall refer to as the **Dehn twist about** a. The following result is remarkable and important:

Theorem 7. *For any surface $\Sigma_{g,n}$, $\Gamma_{g,n}$ is generated by finitely many Dehn twists.*

Furthermore, for $g \geq 0$ and $n \geq 1$, Dehn twists about the following simple closed curves generate $\Gamma_{g,n}$:



6 A Hensel-type argument

Suppose we have polynomials $f_1, ..., f_m \in \mathbb{Z}[x_1, ..., x_n]$. Then for any unital commutative ring R, define

$$X(R) := \{\vec{r} = (r_1, ..., r_n) \in R^n \mid f_i(\vec{r}) = 0 \ \forall i \in \{1, ..., m\}\}.$$

Let $p \geq 3$ be a prime, and let $\mathbb{Z}_p$ denote the p-adic integers. For $k \geq 1$, let $\pi_k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$ denote the residue modulo p^k ring homomorphism. By an abuse of notation, we will often use π_k to also denote the map $(\mathbb{Z}_p)^n \to (\mathbb{Z}/p^k\mathbb{Z})^n$ defined by applying π_k to each entry of $(\mathbb{Z}_p)^n$. Then for a fixed $k \geq 1$ and a fixed $\vec{a} \in (\mathbb{Z}_p)^n$ such that $\pi_k(\vec{a}) \in X(\mathbb{Z}/p^k\mathbb{Z})$, we would like to know for which $\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n$ it is true that $\pi_{k+1}(\vec{a}) + p^k\vec{x} \in X(\mathbb{Z}/p^{k+1}\mathbb{Z})$. That is, we are interested in the set

$$\{\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n \mid \pi_{k+1}(\vec{a}) + p^k\vec{x} \in X(\mathbb{Z}/p^{k+1}\mathbb{Z})\}.$$

This set describes exactly the elements in $X(\mathbb{Z}/p^{k+1}\mathbb{Z})$ which get projected onto the same element $\pi_k(\vec{a}) \in X(\mathbb{Z}/p^k\mathbb{Z})$. We may call these elements the "children" of $\pi_k(\vec{a})$.

For $f \in \mathbb{Z}[x_1, ..., x_n]$ and $\vec{a}, \vec{x} \in (\mathbb{Z}_p)^n$, the **Taylor expansion of f about $\vec{a}$** is given by

$$f(\vec{x}) = \sum_{I=(i_1,...,i_n)\in\mathbb{Z}^n_{\geq 0}} \frac{\partial_I f(\vec{a})}{I!}(\vec{x}-\vec{a})^I$$

where

$I! := i_1! \cdots i_n!$, $\partial_I f := \partial_1^{i_1} \cdots \partial_n^{i_n} f$, and $(\vec{x}-\vec{a})^I := (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$.

Fix $k \geq 1$ and $\vec{a} \in (\mathbb{Z}_p)^n$ such that $\pi_k(\vec{a}) \in X(\mathbb{Z}/p^k\mathbb{Z})$. Then for $\vec{x} \in (\mathbb{Z}_p)^n$ and $j \in \{1, ..., m\}$, we have by Taylor expansion about $\vec{a}$ that

$$f_j(\vec{a} + p^k\vec{x}) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \frac{\partial_I f_j(\vec{a})}{I!}(p^k\vec{x})^I$$

$$= f_j(\vec{a}) + p^k \sum_{i=1}^n \partial_i f_j(\vec{a})x_i + \sum_{I\in\mathbb{Z}^n_{\geq 0}, |I|\geq 2} \frac{p^{k|I|}\partial_I f_j(\vec{a})}{I!}\vec{x}^I.$$

It is a basic fact from number theory that

$$\nu_p(i!) = \lfloor \frac{i}{p} \rfloor + \lfloor \frac{i}{p^2} \rfloor + \cdots .$$

Therefore for $i \geq 1$,

$$\nu_p(i!) < \frac{i}{p} + \frac{i}{p^2} + \cdots$$

$$= \frac{i}{p}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots) = \frac{i}{p}(\frac{1}{1-\frac{1}{p}}) = \frac{i}{p-1}.$$

Notice that for $k \geq 1$, $|I| \geq 2$, and $p \geq 3$,

$$k|I| - \frac{|I|}{p-1} = |I|(k - \frac{1}{p-1}) \geq 2(k - \frac{1}{p-1}) = 2k - \frac{2}{p-1} \geq 2k - 1 \geq k.$$

So for $k \geq 1$, $|I| \geq 2$, and $p \geq 3$,

$$\nu_p(\frac{p^{k|I|}}{I!}) = \nu_p(p^{k|I|}) - \nu_p(I!) > k|I| - \frac{|I|}{p-1} \geq k,$$

which means that

$$\nu_p(\frac{p^{k|I|}}{I!}) \geq k+1.$$

Thus for $\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n$,

$$f_j(\pi_{k+1}(\vec{a}) + p^k\vec{x}) = f_j(\pi_{k+1}(\vec{a})) + p^k \sum_{i=1}^n \partial_i f_j(\pi_{k+1}(\vec{a}))x_i.$$

Observe that $f_j(\pi_k(\vec{a})) = 0$ implies that there exists $t_j \in \mathbb{Z}_p$ such that $f_j(\vec{a}) = p^k t_j$. Hence

$$f_j(\pi_{k+1}(\vec{a}) + p^k \vec{x}) = 0 \;\; \forall j \iff p^k \pi_{k+1}(t_j) + p^k \sum_{i=1}^{n} \partial_i f_j(\pi_{k+1}(\vec{a}))x_i = 0 \;\; \forall j$$

$$\iff \pi_1(t_j) + \sum_{i=1}^{n} \partial_i f_j(\pi_1(\vec{a}))\pi_1(x_i) = 0 \;\; \forall j$$

$$\iff \begin{bmatrix} \partial_1 f_1(\pi_1(\vec{a})) & \cdots & \partial_n f_1(\pi_1(\vec{a})) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \partial_1 f_m(\pi_1(\vec{a})) & \cdots & \partial_n f_m(\pi_1(\vec{a})) \end{bmatrix} \begin{bmatrix} \pi_1(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \pi_1(x_n) \end{bmatrix} = - \begin{bmatrix} \pi_1(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \pi_1(t_m) \end{bmatrix}. \quad (*)$$

Define the Jacobian of $f_1, ..., f_m$ by

$$J(f_1, ..., f_m) := [\partial_j f_i] \in \mathrm{M}_{m \times r}(\mathbb{Z}[x_1, ..., x_r]).$$

For A a unital commutative ring and $\mathbf{a} \in A^r$, define

$$J(f_1, ..., f_m)(\mathbf{a}) := [\partial_j f_i(\mathbf{a})] \in \mathrm{M}_{m \times r}(A).$$

What we have shown then is that if $J(f_1, ..., f_m)(\pi_1(\vec{a}))$ is full rank and $m \leq n$, then the set

$$\{\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n \mid \pi_{k+1}(\vec{a}) + p^k \vec{x} \in X(\mathbb{Z}/p^{k+1}\mathbb{Z})\}$$

is non-empty, and by the rank-nullity theorem, has p^{n-m} elements.

Moreover, let $\mathrm{T}_{\pi_1(\vec{a})} X(\mathbb{Z}/p\mathbb{Z})$ denote the kernel of the map $J(f_1, ..., f_m)(\pi_1(\vec{a}))$. Then we have also shown that the set

$$\{\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n \mid \pi_1(\vec{x}) \in \mathrm{T}_{\pi_1(\vec{a})} X(\mathbb{Z}/p\mathbb{Z})\}$$

differs from the set

$$\{\vec{x} \in (\mathbb{Z}/p^{k+1}\mathbb{Z})^n \mid \pi_{k+1}(\vec{a}) + p^k \vec{x} \in X(\mathbb{Z}/p^{k+1}\mathbb{Z})\}$$

by a translation.

7 Progress with twice-punctured torus

The example we have worked with so far has been the twice-punctured torus, that is, $g = 1$ and $n = 2$, and we have chosen $p = 13$, $\epsilon = \varnothing$, and $\mathbf{k} = (-2, 2)$. Our over-arching goal then is to show that the number of $\Gamma_{1,2}$-orbits in $\mathrm{C}^{\bullet}_{1,2;(-2,2)}(\mathbb{Z}/13^k\mathbb{Z})$ is constant for increasing $k \geq 1$. However, since the definition of $\mathrm{C}^{\bullet}_{1,2;(-2,2)}(\mathbb{Z}/13^k\mathbb{Z})$ is a bit technical, our strategy has been to investigate the $\Gamma_{1,2}$-orbits of a different but related set which is more straightforward to compute with, using the results of the last section. Then we expect to be able to expand our argument to all of $\mathrm{C}^{\bullet}_{1,2;(-2,2)}(\mathbb{Z}/13^k\mathbb{Z})$.

Recall our notation from Section 1 that

$$\Pi_{1,2} \cong \langle a, b, c_1, c_2 \mid [a, b^{-1}]c_1c_2\rangle.$$

Combining the result of Horowitz and our example from Section 2, we have that $\mathrm{Ch}_{1,2}$ is defined by the equation

$$\begin{array}{c} t_a^2 + t_b^2 + t_{c_1}^2 + t_{ab}^2 + t_{ac_1}^2 + t_{bc_1}^2 + t_{abc_1}^2 - (t_a t_b t_{ab} + t_{c_1} t_{ab} t_{abc_1} \\ +t_b t_{c_1} t_{bc_1} + t_a t_{bc_1} t_{abc_1} + t_a t_{c_1} t_{ac_1} + t_b t_{ac_1} t_{abc_1}) \\ +t_{ab} t_{ac_1} t_{bc_1} + t_a t_b t_{c_1} t_{abc_1} - 4 = 0 \end{array}.$$

Using the Fricke identities, we can deduce that

$$t_{c_2} = t_a t_{ac_1} + t_b t_{bc_1} + t_{ab} t_{abc_1} - t_a t_b t_{abc_1} - t_{c_1}.$$

We want to investigate what happens when we fix the values of t_{c_1} and t_{c_2} to be -2 and 2.

Thus, define

$$f_1 := \sum_{i=1}^{6} T_i^2 + 2T_1T_3 - T_2T_3T_6 - T_1T_2T_5 + 2T_5T_6 + 2T_2T_4 - T_1T_4T_6 + T_3T_4T_5 - 2T_1T_2T_6$$

and

$$f_2 := T_1T_3 + T_2T_4 + T_5T_6 - T_1T_2T_6.$$

T_1 through T_6 correspond to the generators of $R(\Gamma_{1,2}, \mathrm{SL}_2)$ given the ordering $a < b < c_1$.

In the spirit of the last section, for $k \geq 1$, define

$$X(\mathbb{Z}/13^k\mathbb{Z}) := \{\vec{a} \in (\mathbb{Z}/13^k\mathbb{Z})^6 \mid f_1(\vec{a}), f_2(\vec{a}) = 0\}.$$

The action of $\Gamma_{1,2}$ is well-defined on $X(\mathbb{Z}/13^k\mathbb{Z})$, so it still makes sense to investigate orbit stability in this context. In particular we want to look at $X(\mathbb{Z}/13\mathbb{Z})$ and $X(\mathbb{Z}/13^2\mathbb{Z})$. However, instead of looking at all the elements of $X(\mathbb{Z}/13\mathbb{Z})$ at once, we will fix a single element $\vec{x} \in X(\mathbb{Z}/13\mathbb{Z})$ and look at its children, and we will check if $\mathrm{Stab}_{\Gamma_{1,2}}(\vec{x})$ acts transitively on the children.

Utilizing Section 5, we see that $\Gamma_{1,2}$ is generated by Dehn twists about the simple loops homotopic to a, b, and c_1. Recall that the action of an element γ of $\Gamma_{1,2}$ is given by choosing an element of $\mathrm{Aut}(\Pi_{1,2})$ which represents the element of $\mathrm{Out}(\Pi_{1,2})$ associated to γ. Golsefidy and Tamam show in Section

8.2 of [14] that the following automorphisms of $\Pi_{1,2}$ are suitable representative elements of the generating Dehn twists:

$$\tau_1(a, b, c_1) := (a, ab, c_1)$$

$$\tau_2(a, b, c_1) := (ab^{-1}, b, c_1)$$

$$\tau_3(a, b, c_1) := (a, bd^{-1}, dc_1 d^{-1}),$$

where $d := c_1^{-1} b^{-1} ab$.

Using these Dehn twist "lifts", along with the Fricke identities, we can look at how the Dehn twists act on the generating set of $R(\Pi_{1,2}, \mathrm{SL}_2)$. This will give an explicit description of the action of $\Gamma_{1,2}$ on $X(\mathbb{Z}/13^k\mathbb{Z})$ as an action on $(\mathbb{Z}/13^k\mathbb{Z})^6$.

We are currently in the process of calculating these explicit descriptions and writing a program in Mathematica to check if these descriptions lead to a transitive action of $\mathrm{Stab}_{\Gamma_{1,2}}(\vec{x})$ on the children of a carefully selected solution.

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